

$f(R)$: a tale of two frames

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to IOAV WAGA!



General Relativity (GR)

$$S = \int \sqrt{-g} f(R) d^4x$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$$

$$R = 0$$

Jordan frame

$$S = \int \sqrt{-g} f(R) d^4x$$

$$\begin{aligned} f_R \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) &= \\ &= \frac{1}{2} g_{\mu\nu} (f - R f_R) + \nabla_\mu \nabla_\nu f_R - \\ &- g_{\mu\nu} \nabla_\alpha \nabla^\alpha f_R \end{aligned}$$

Jordan frame

$$S = \int \sqrt{-g} f(R) d^4x$$

$$f_R \equiv \frac{df}{dR}$$

$$f(GR) = R$$

$$f_R(GR) = 1$$

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$$3\Box f_R(R) + R \cdot f_R(R) - 2f(R) = 0$$

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$$\boxed{3\Box f_R(R) + R \cdot f_R(R) - 2f(R) = 0}$$

Stability conditions

Pogosian and Silvestri, arxiv:0709.0296v2

$$f(R) = R + \Delta(R)$$

$$\lim_{R \rightarrow \infty} \frac{\Delta}{R} = 0$$

GR @ early times

$$f_R \equiv 1 + \Delta_R > 0$$

no ghosts

$$f_{RR} \equiv \Delta_{RR} > 0$$

no tachyons

Stability conditions

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no tachyons

Sotiriou and Faraoni, arxiv:0805.1726v2

$$f(R) = R + \epsilon \Delta(R)$$

$$\square R + \frac{\Delta_{RRR}}{\Delta_{RR}} \nabla^\alpha R \nabla_\alpha R + \frac{\epsilon \Delta_R - 1}{3\epsilon \Delta_{RR}} R = \frac{-1}{3\epsilon \Delta_{RR}} T + \frac{2\Delta}{3\Delta_{RR}}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$R = T + R_1$$

< 0 : *tachyons* !

$$\ddot{R}_1 - \nabla^2 R_1 - \frac{2\Delta_{RRR}}{\Delta_{RR}} \dot{T} \dot{R}_1 - \frac{1}{3\Delta_{RR}} \left(\frac{1}{\epsilon} - \Delta_R \right) R_1 +$$

$$+ \frac{2\Delta_{RRR}}{\Delta_{RR}} \vec{\nabla} T \cdot \vec{\nabla} R_1 = \ddot{T} - \nabla^2 T - \frac{T\Delta_R + \Delta}{3\Delta_{RR}}$$

$$f_{RR} \equiv \Delta_{RR} > 0$$

A change of frames...

Conformal transformation

to Einstein frame:

$$\tilde{g}_{\mu\nu} \equiv \chi g_{\mu\nu}$$

$$\chi \equiv f' = \exp \left[\sqrt{\frac{2}{3}} \tilde{\chi} \right]$$

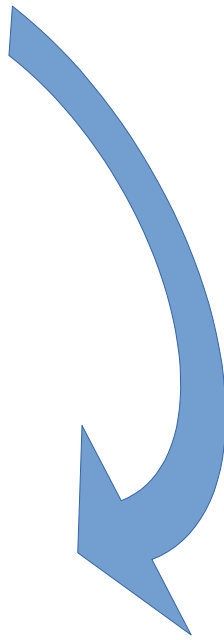
$$S = \int d^4x \sqrt{-g} \frac{1}{2} \left[\tilde{R} - \tilde{g}^{\mu\nu} \partial_\mu \tilde{\chi} \partial_\nu \tilde{\chi} - 2V(\tilde{\chi}) \right] +$$

$$+ S_{(m,r)}[\tilde{g}_{\mu\nu}, \psi_a]$$

$$V(\tilde{\chi}) \equiv \frac{Rf' - f}{2(f')^2}$$

Jordan frame

$$S = \int \sqrt{-g} f(R) d^4x$$



Einstein frame

$$S = \int d^4x \sqrt{-g} \frac{1}{2} \left[\tilde{R} - \tilde{g}^{\mu\nu} \partial_\mu \tilde{\chi} \partial_\nu \tilde{\chi} - 2V(\tilde{\chi}) \right]$$

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Jordan frame

?

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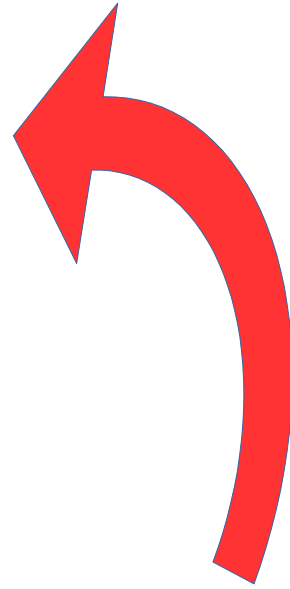
Einstein frame

$$S = \int d^4x \sqrt{-g} \frac{1}{2} \left[\tilde{R} - \tilde{g}^{\mu\nu} \partial_\mu \tilde{\chi} \partial_\nu \tilde{\chi} - 2V(\tilde{\chi}) \right]$$

Jordan frame

?

$$S = \int \sqrt{-g} f(R) d^4x$$



$$\tilde{g}_{\mu\nu} \equiv \chi g_{\mu\nu}$$

$$\chi \equiv f_R = \exp \left[\sqrt{\frac{2}{3}} \tilde{\chi} \right]$$

$$V(\tilde{\chi}) = \frac{1}{2} \tilde{\chi}^2$$

Einstein frame

$$S = \int d^4x \sqrt{-g} \frac{1}{2} \left[\tilde{R} - \tilde{g}^{\mu\nu} \partial_\mu \tilde{\chi} \partial_\nu \tilde{\chi} - 2V(\tilde{\chi}) \right]$$

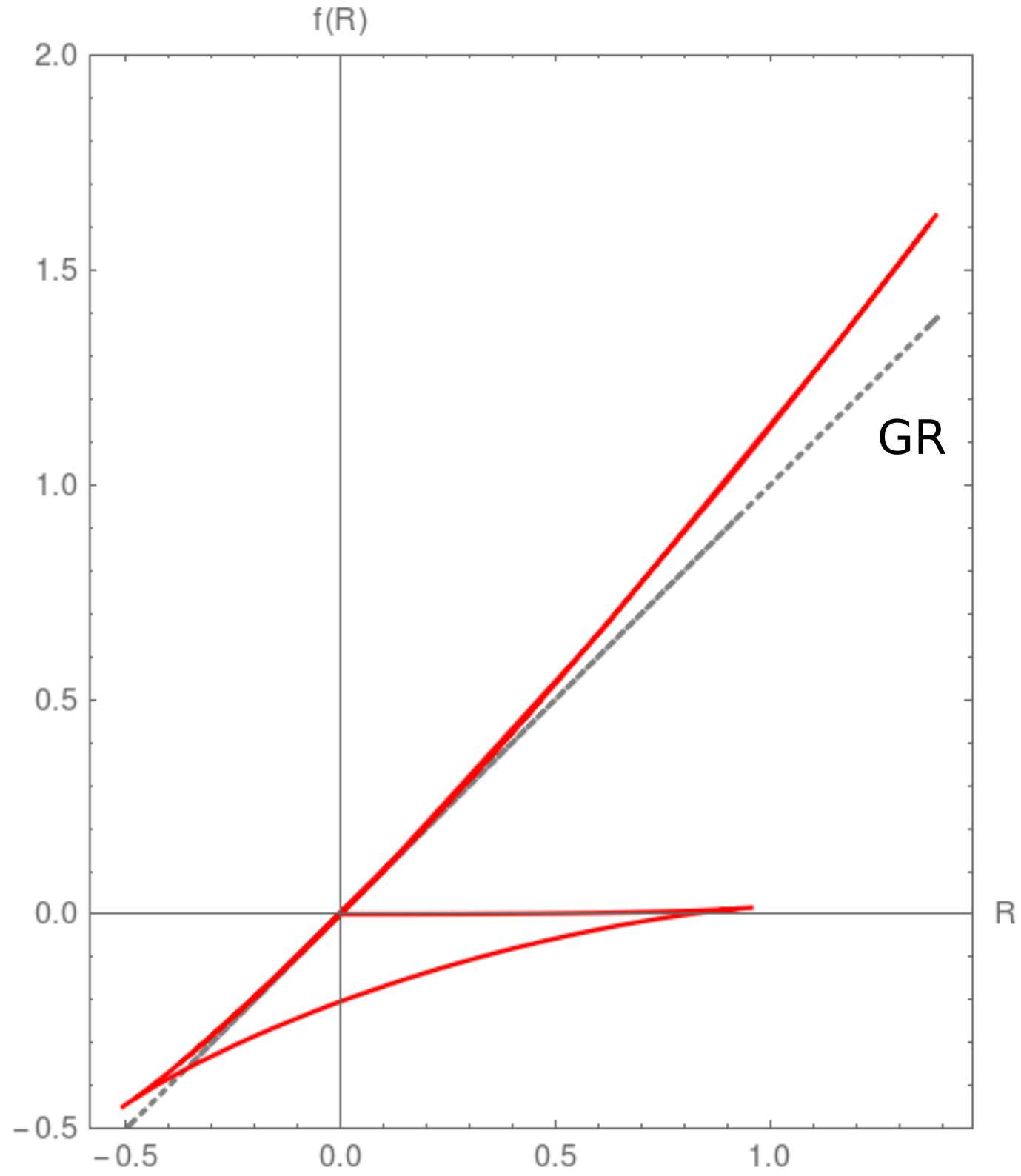
Guido Magnano and Leszek M. Sokokowski,
Phys. Rev. D50 (8), 5039 (1994)

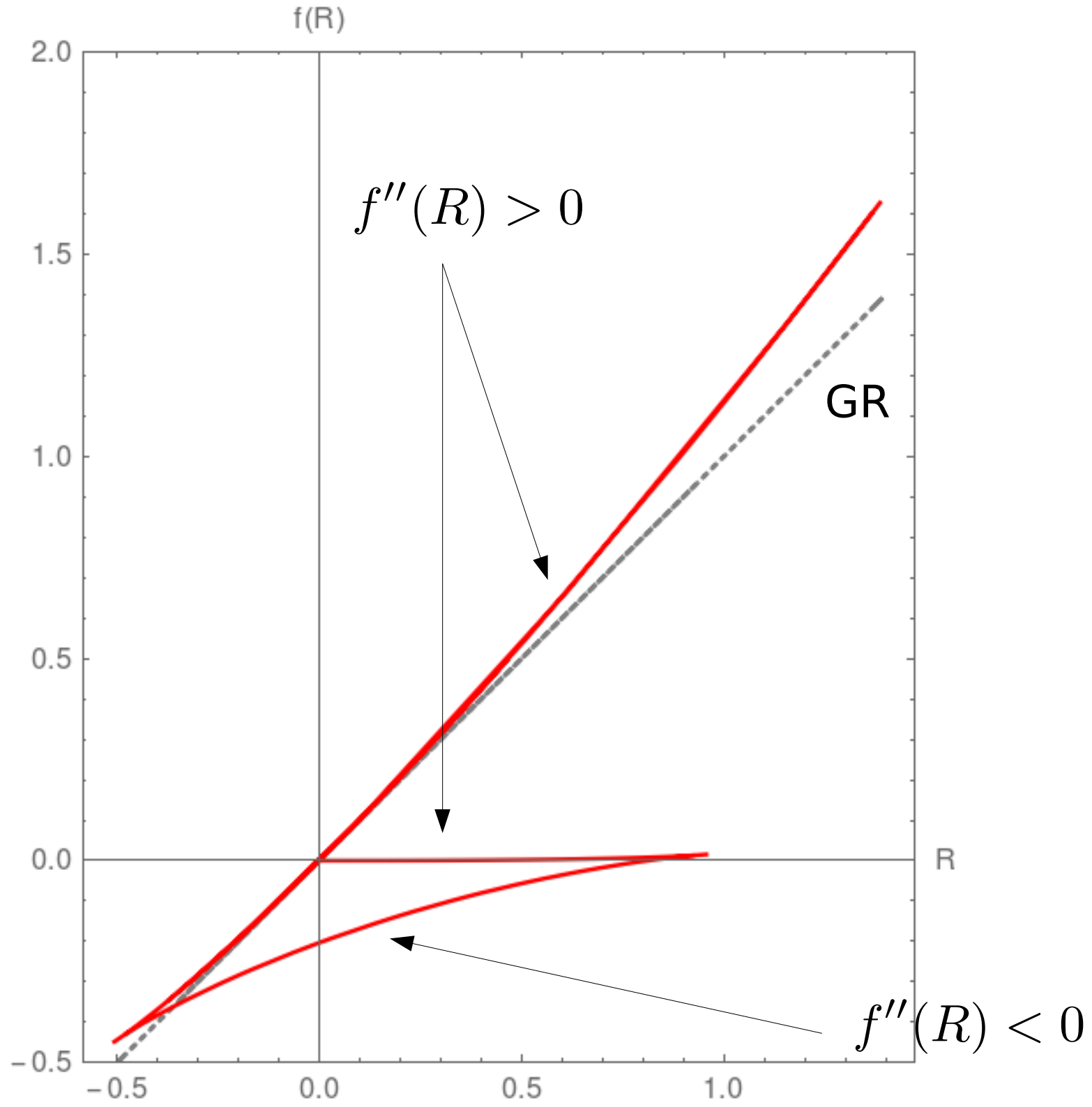
$$V(\tilde{\chi}) = \frac{1}{2} m \tilde{\chi}^n$$

$$f[R(\chi)] = 2\mu \chi^2 (\ln \chi + n) (\ln \chi)^{n-1}$$

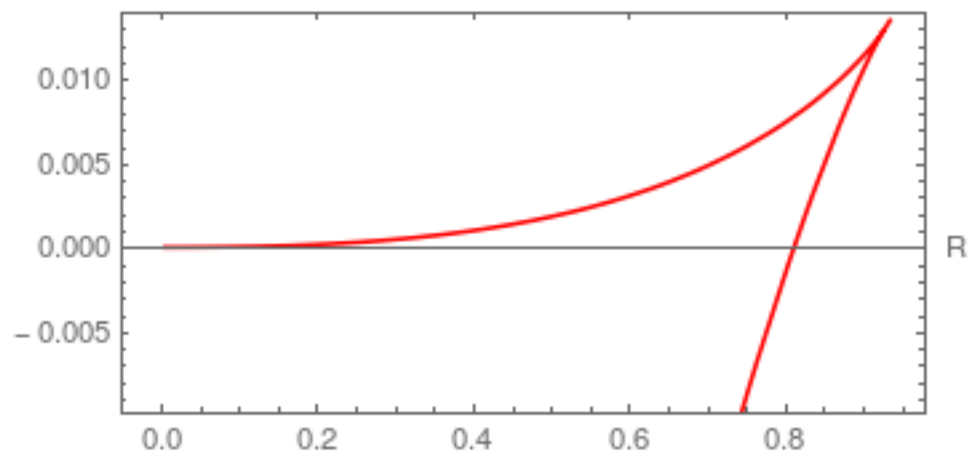
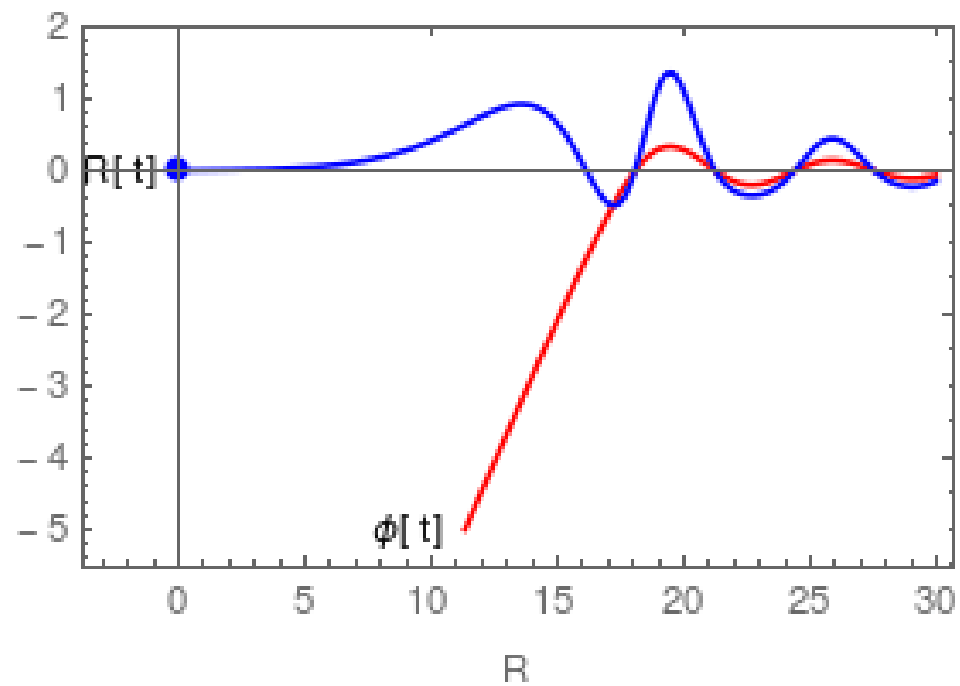
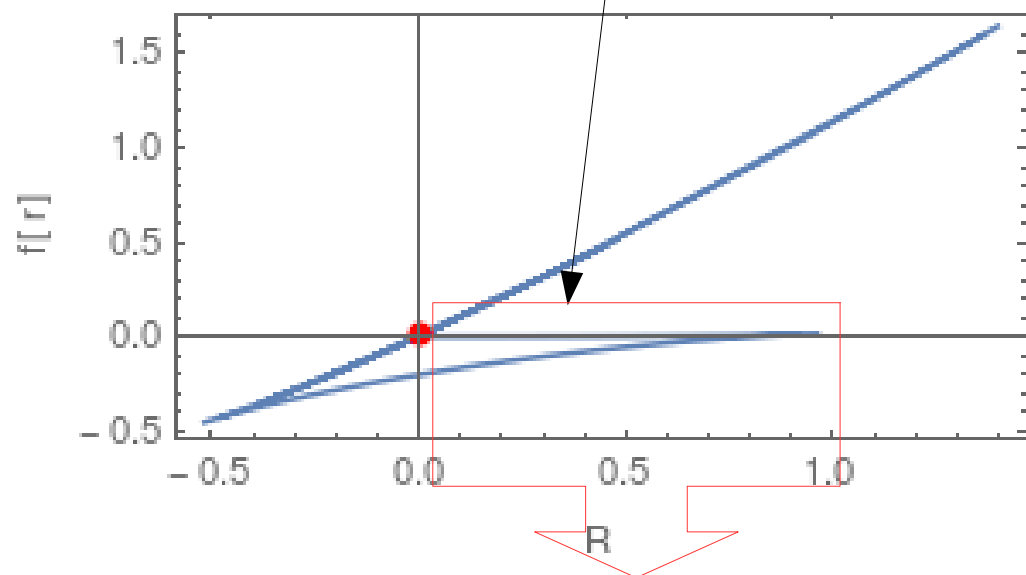
$$R(\chi) = 2\mu \chi (2 \ln \chi + n) (\ln \chi)^{n-1}$$

$$\mu := \frac{1}{2} \left(\frac{3}{2} \right)^{\frac{n}{2}} m$$

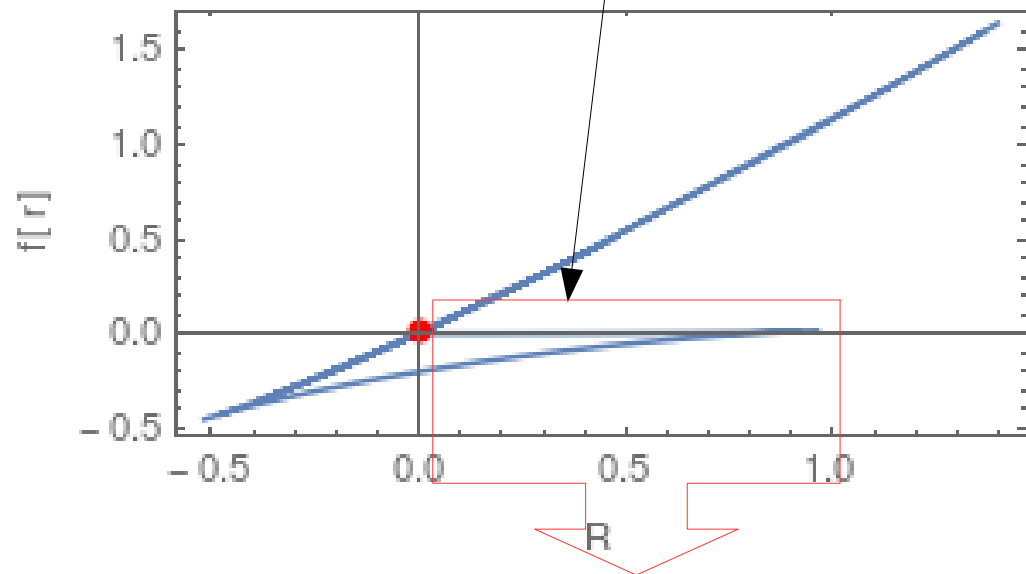




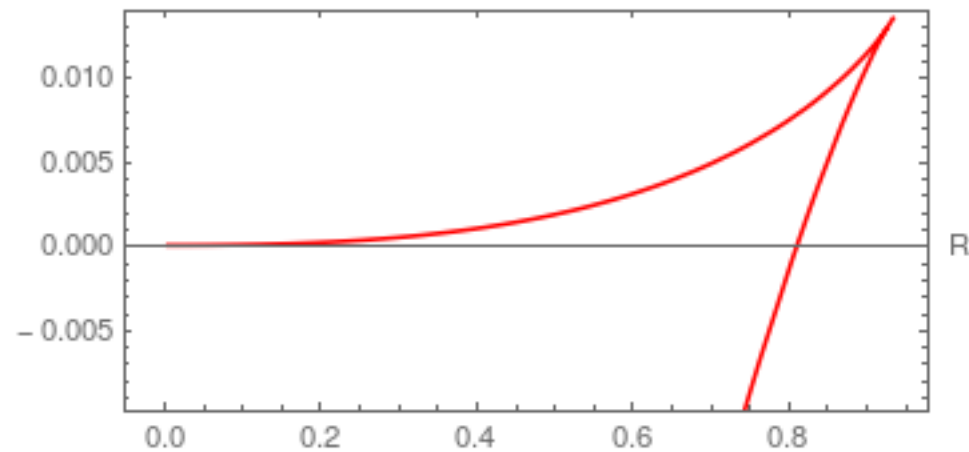
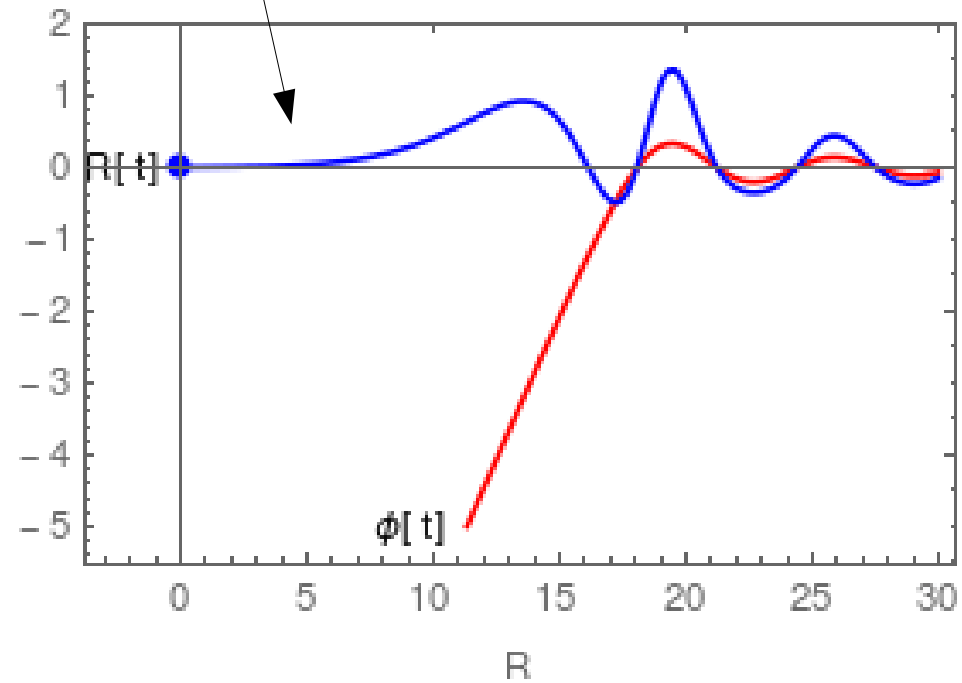
$$f''(R) > 0$$



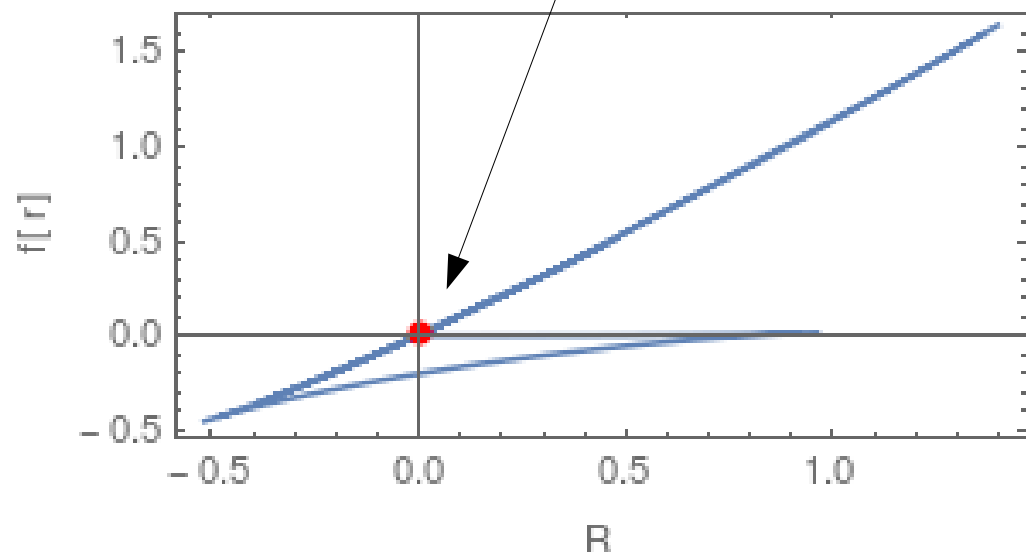
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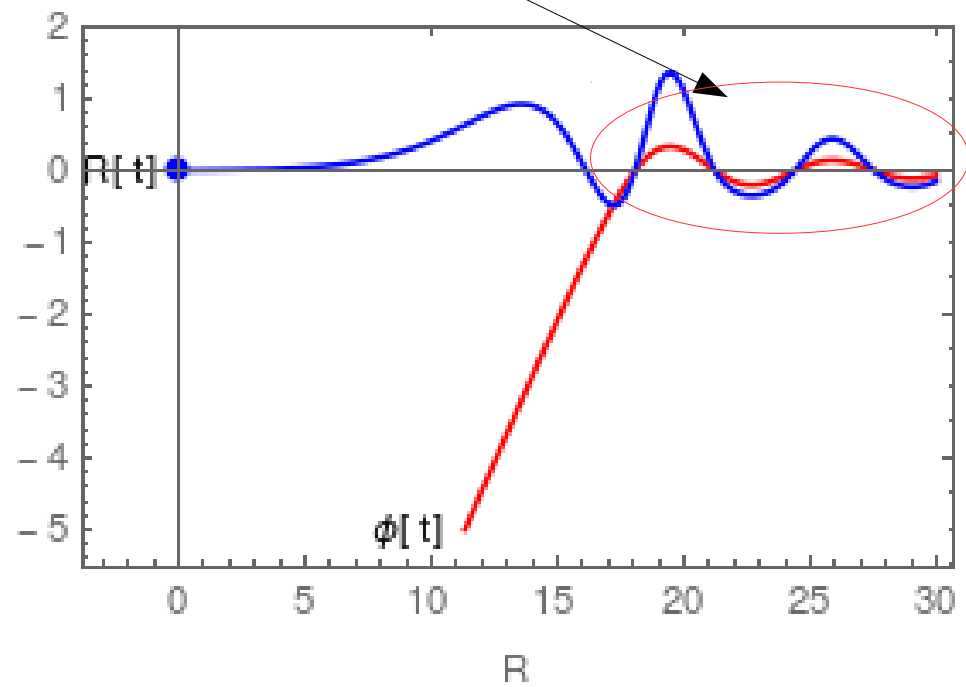
$R = 0$
UNSTABLE
 (SLOW ROLL)



$$f''(R) > 0$$



$R = 0$
STABLE



$$3\Box f_R(R) + R \cdot f_R(R) - 2f(R) = 0$$

$$3\Box f_R(R) + R \cdot f_R(R) - 2f(R) = 0$$

$$\Box = -\frac{\partial^2}{\partial t^2} + \nabla^2$$

$$R = R(t)$$

$$3\Box f_R(R) + R \cdot f_R(R) - 2f(R) = 0$$

$$\Box = -\frac{\partial^2}{\partial t^2} + \nabla^2$$

$$R = R(t)$$

$$\Box R(t) + \frac{f_{RRR}}{3f_{RR}}\partial_\mu R \cdot \partial^\mu R + \frac{f_R(R) \cdot R - 2f(R)}{3f_{RR}(R)} = 0$$

FRW

$$g_{\mu\nu} = \text{diag}(-1, a^2(t), a^2(t)r^2, a^2(t)r^2 \sin^2 \theta)$$

$$\square R(t) + \frac{f_{RRR}}{3f_{RR}} \partial_\mu R \cdot \partial^\mu R + \frac{f_R(R) \cdot R - 2f(R)}{3f_{RR}(R)} = 0$$

$$\square R(t) = 3a^2 \dot{a} \dot{R} + \ddot{R}$$

FRW

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$$b\dot{R}^2 + 3a^2 \dot{a} \dot{R} + \ddot{R} + \frac{dV}{dR} = 0$$

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$$b\dot{R}^2 + 3a^2 \dot{a} \dot{R} + \ddot{R} + \frac{dV}{dR} = 0$$

friction

mass

$$\frac{dV}{dR} \equiv \frac{f_R R - 2f}{3f_{RR}}$$

$$\frac{dV}{dR} \equiv \frac{f_R R - 2f}{3f_{RR}}$$

$$V(R) = \int \frac{f_R R - 2f}{3f_{RR}} dR$$

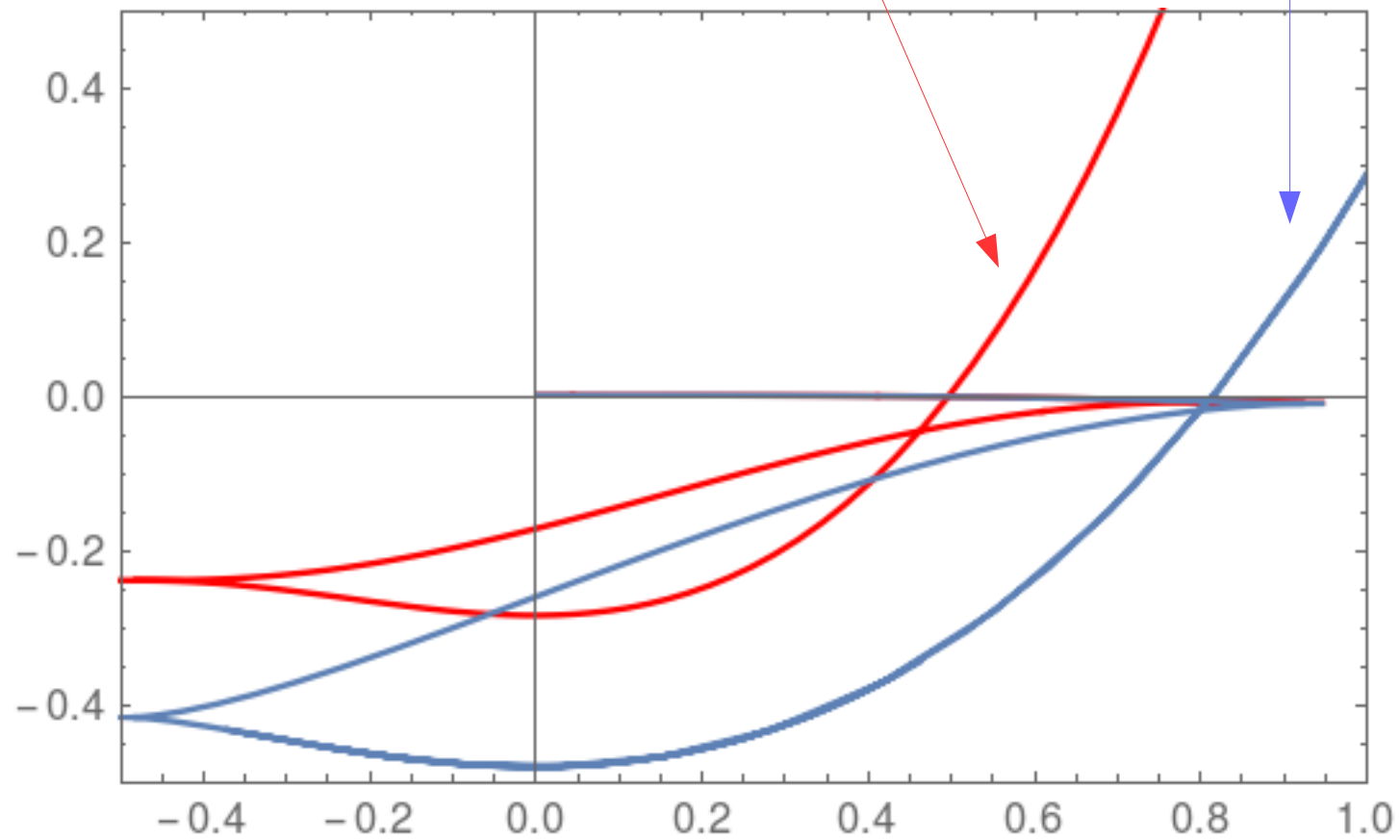
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$$V(R) = \int \frac{f_R R - 2f}{3f_{RR}} \frac{dR}{dt} dt$$

$V[R]$

slow roll

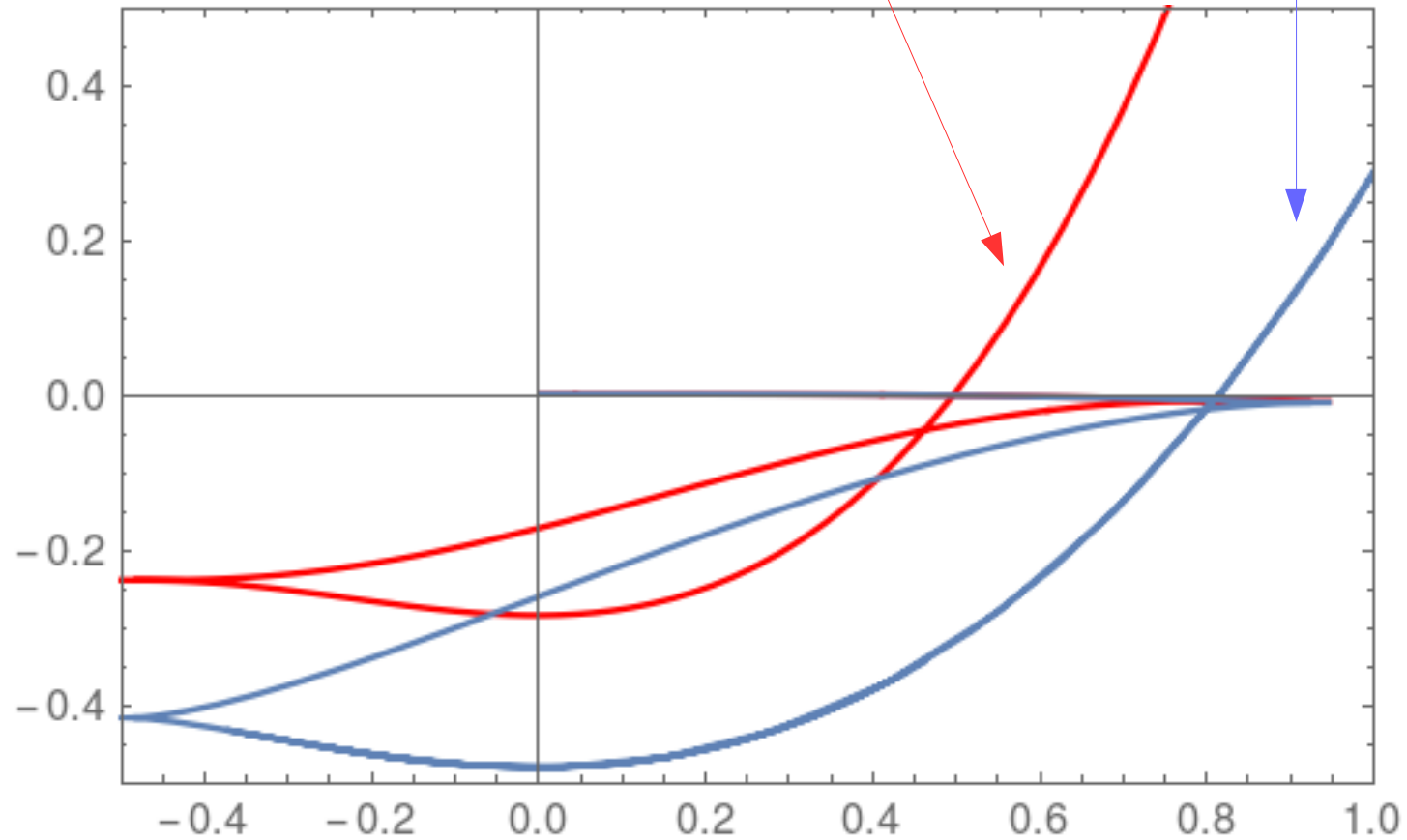
numerical



R

$V[R]$

slow roll **numerical**



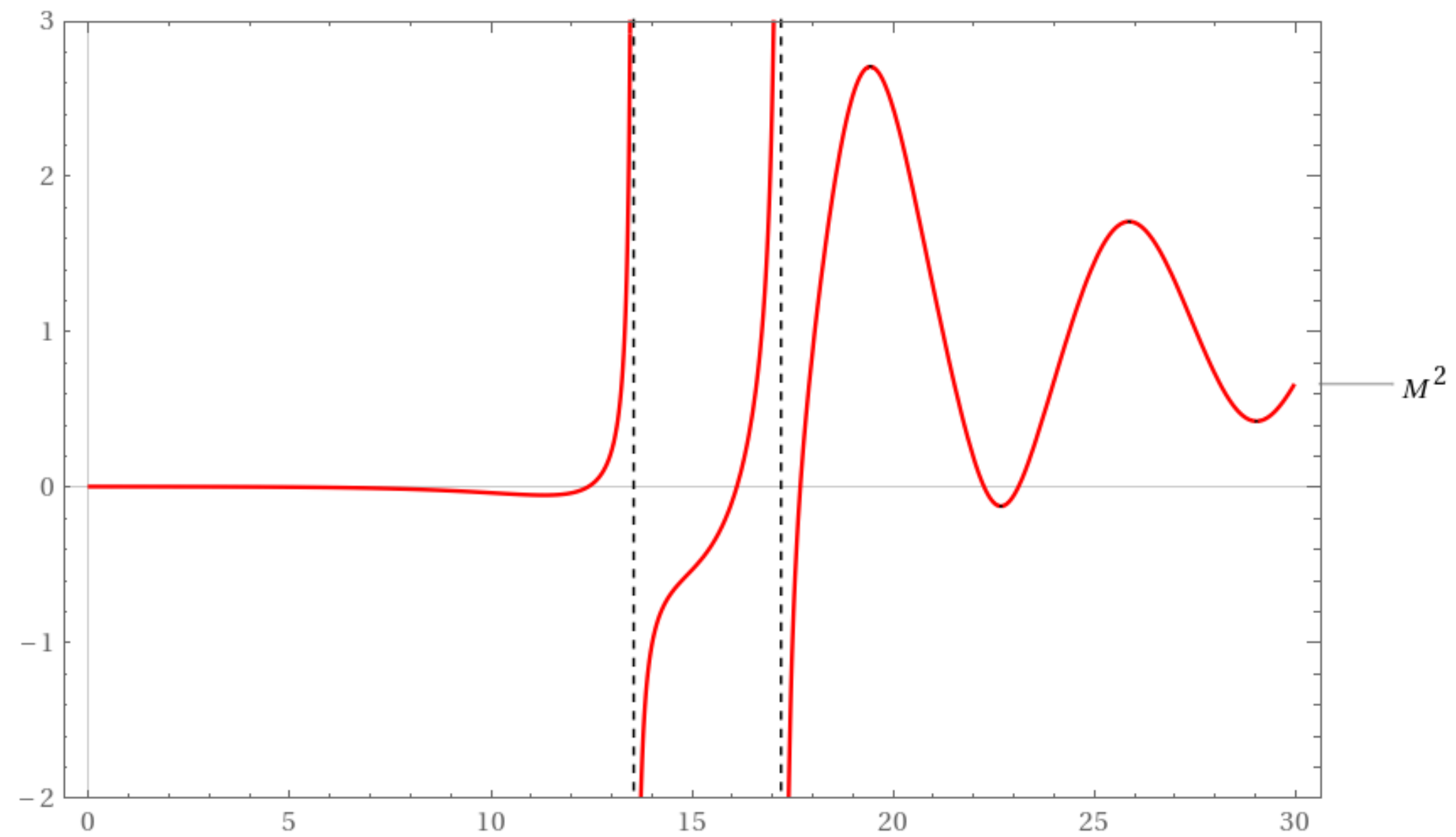
R

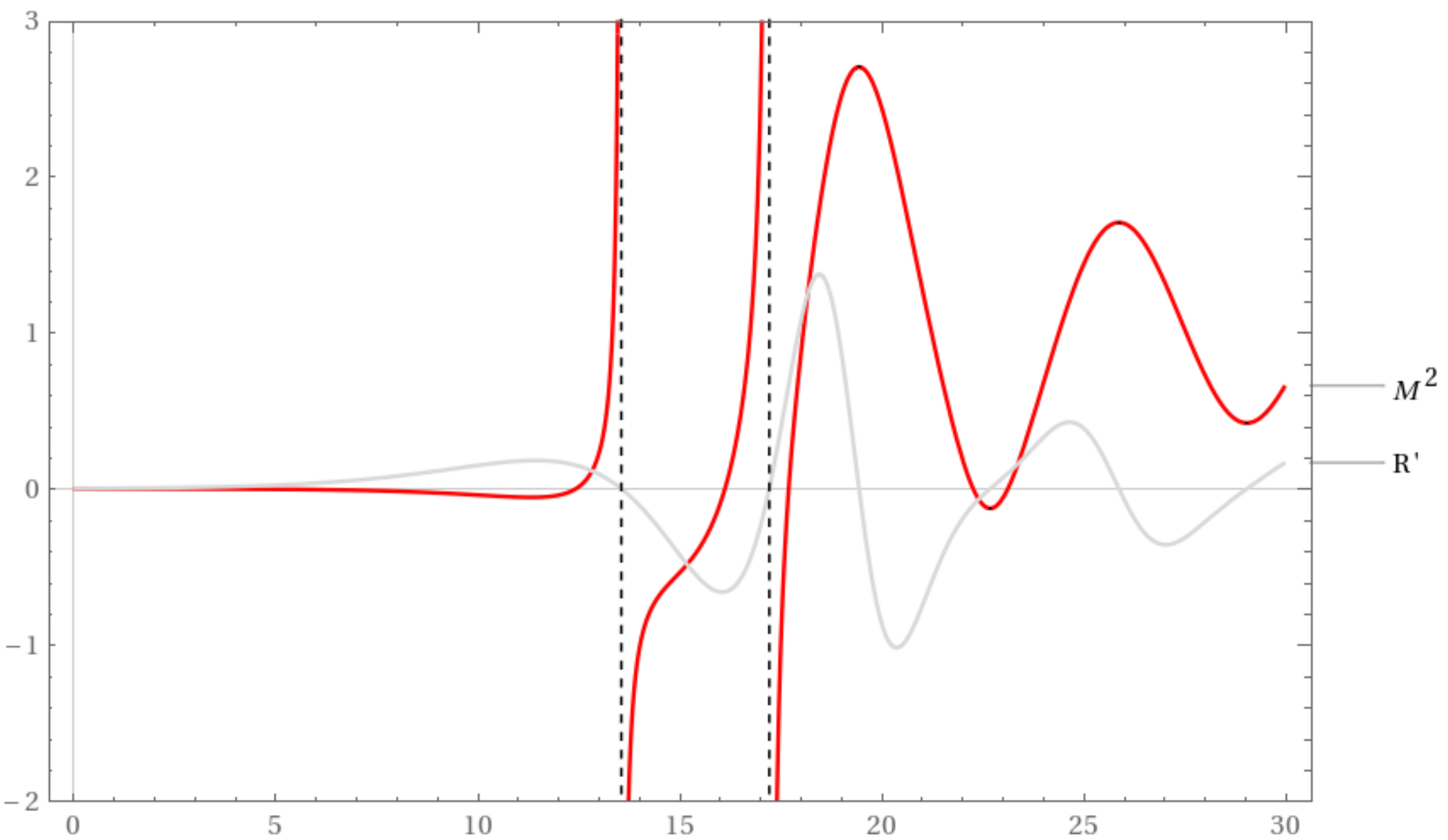
DYNAMICAL CHARACTER !

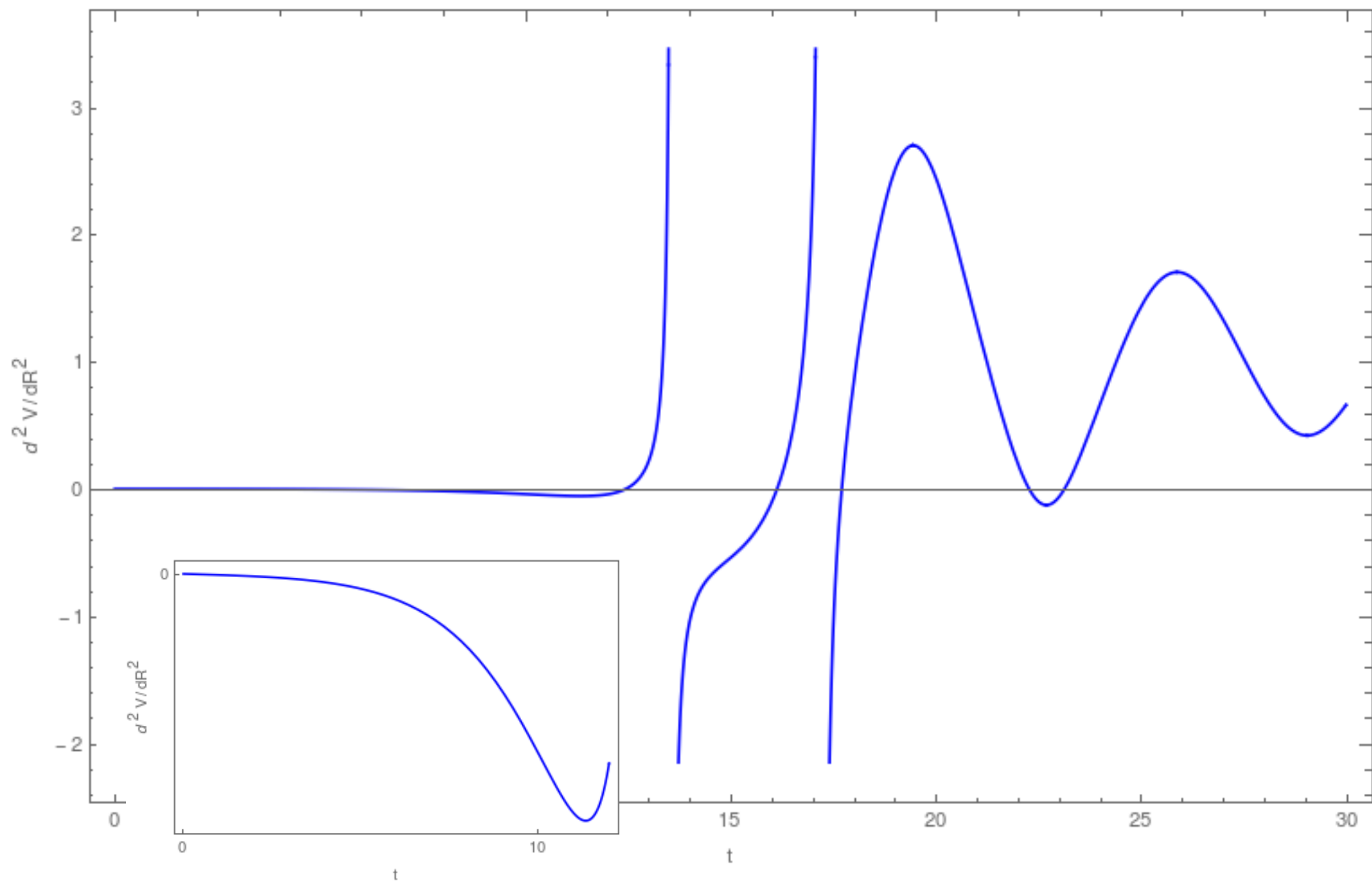
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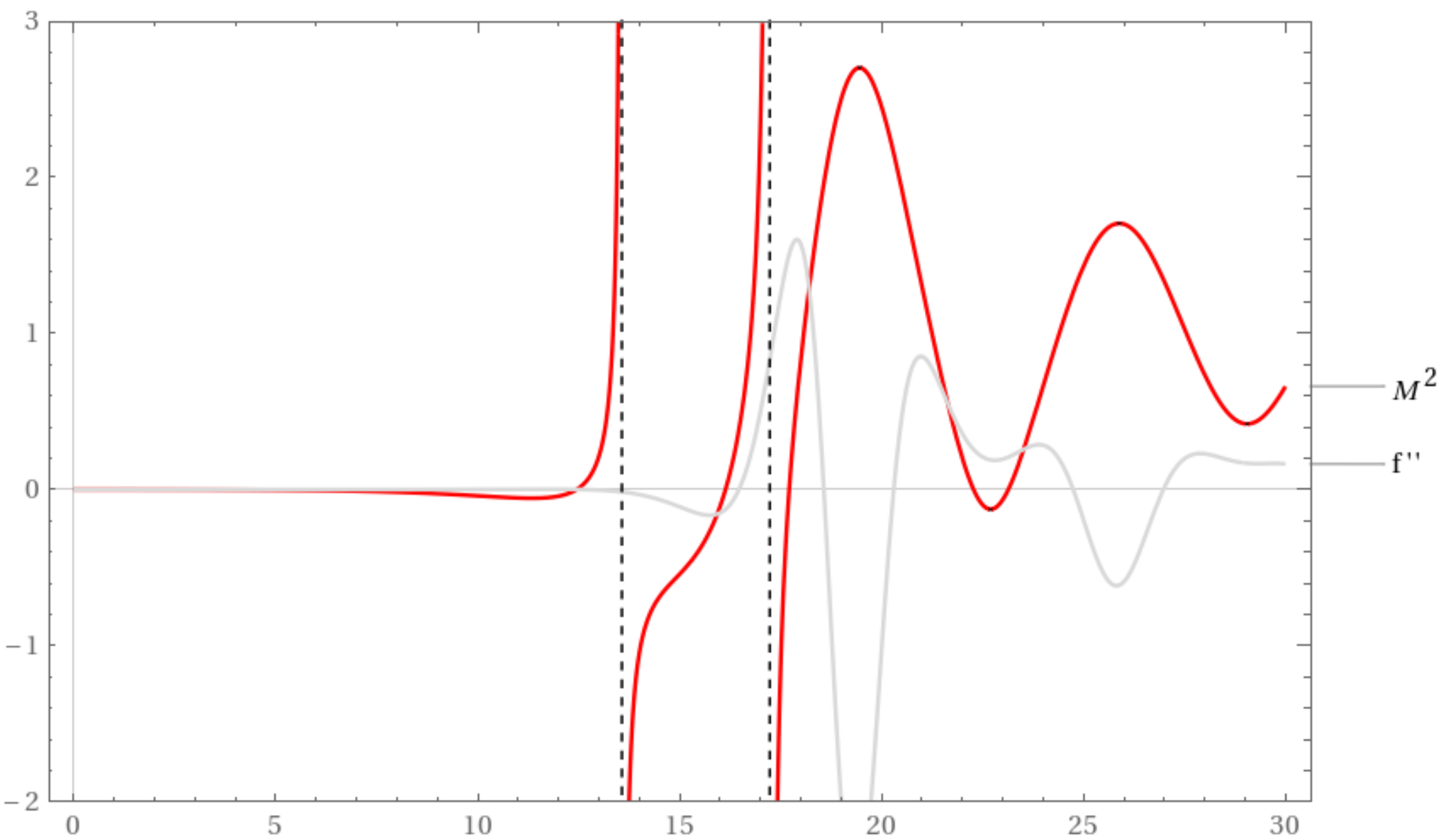
$$V(R) = \int \frac{f_R R - 2f}{3f_{RR}} \frac{dR}{dt} dt$$

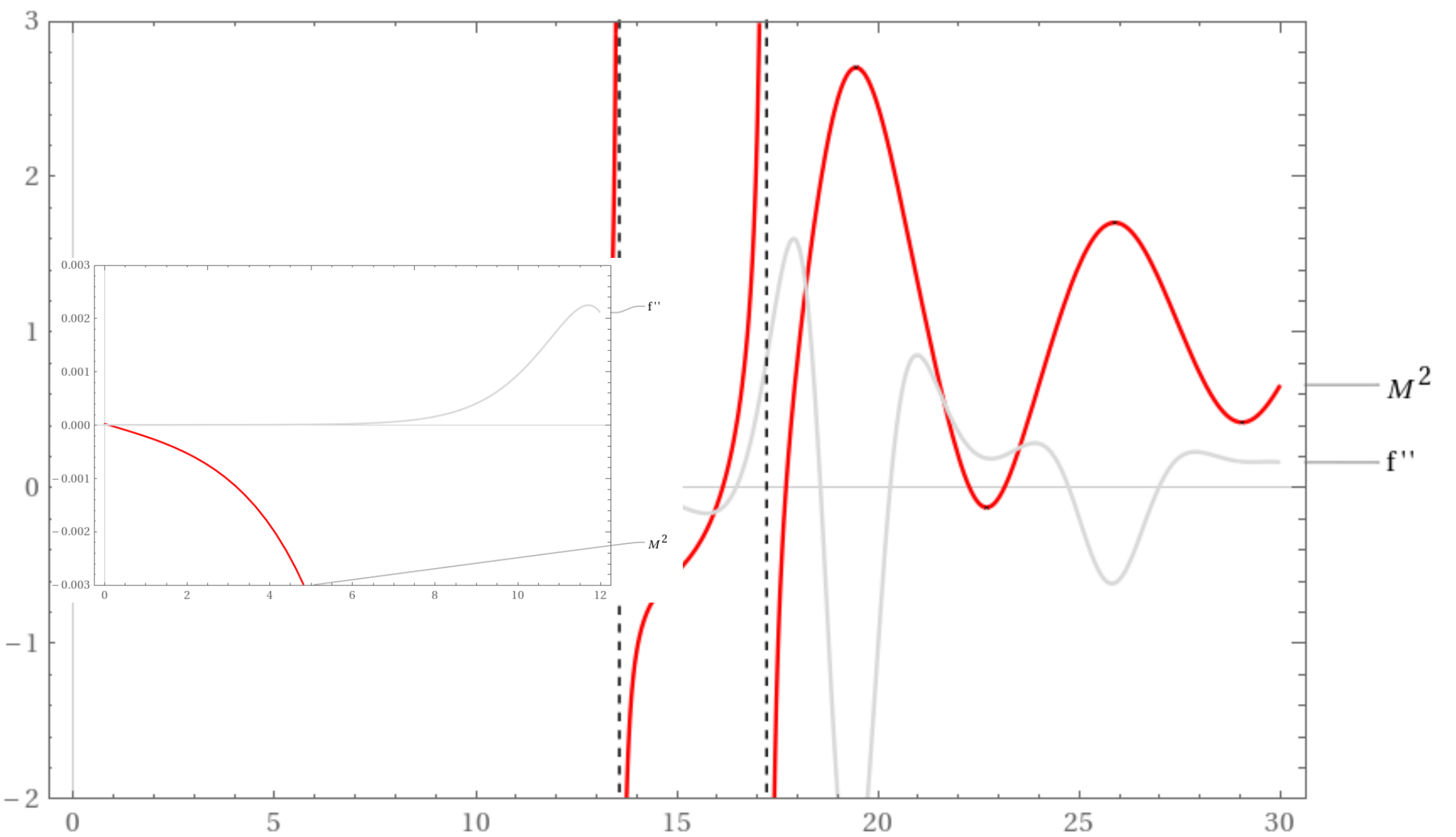
$$M_{eff}^2 \equiv \frac{d^2 V}{dR^2} = \frac{f_{RRR}(2f - Rf_R) - f_{RR}f_R}{3f_{RR}^2} + \frac{1}{3}R$$

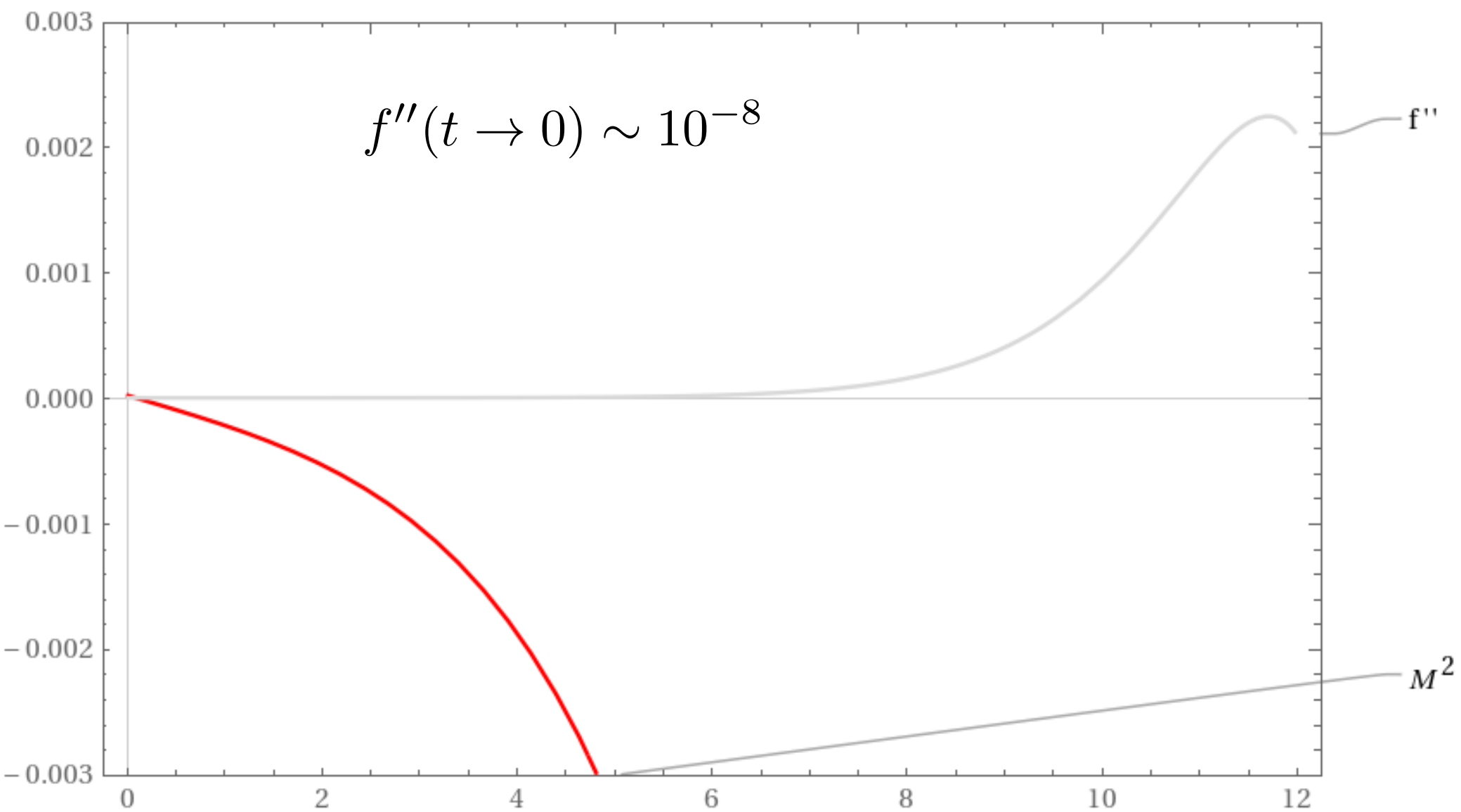


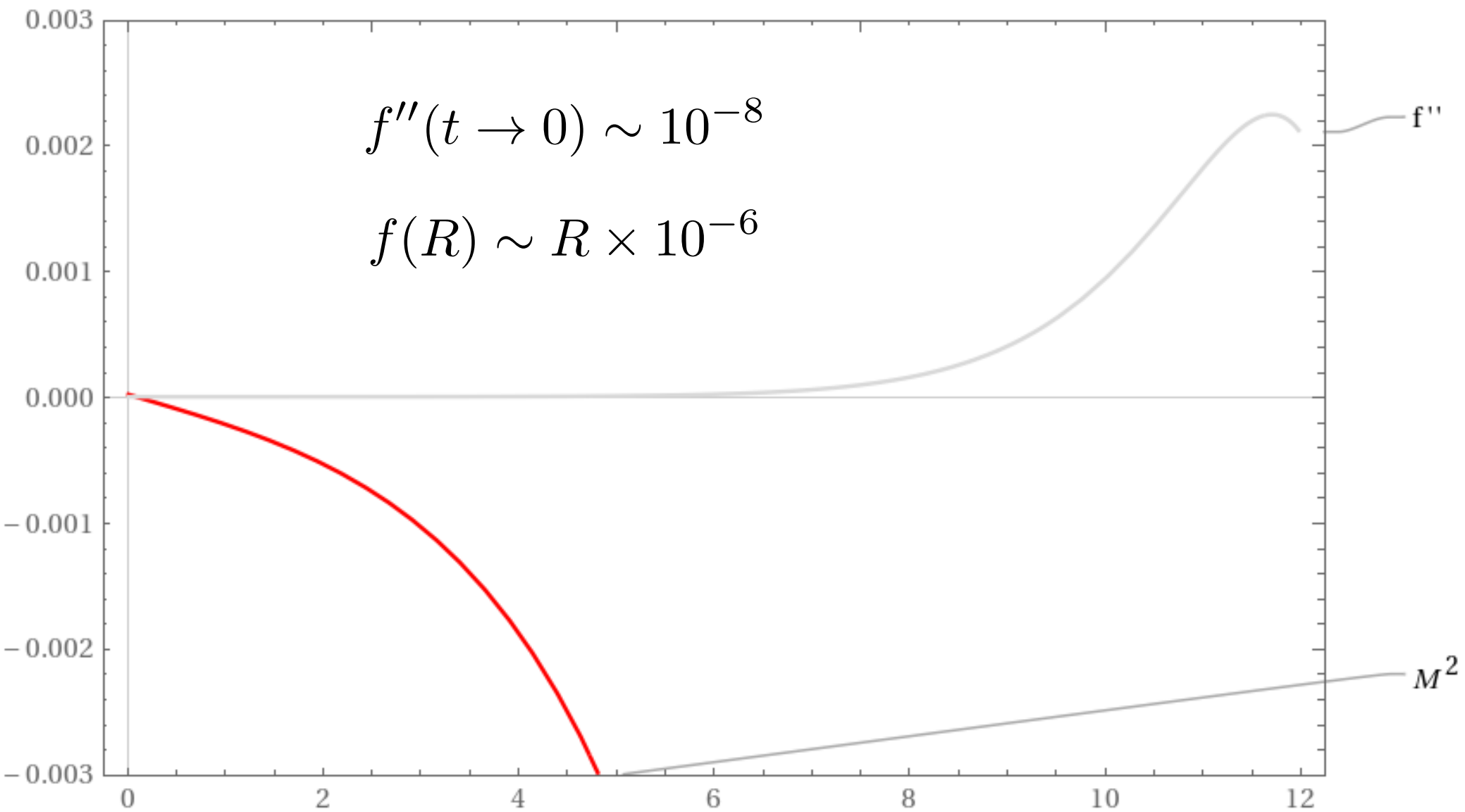


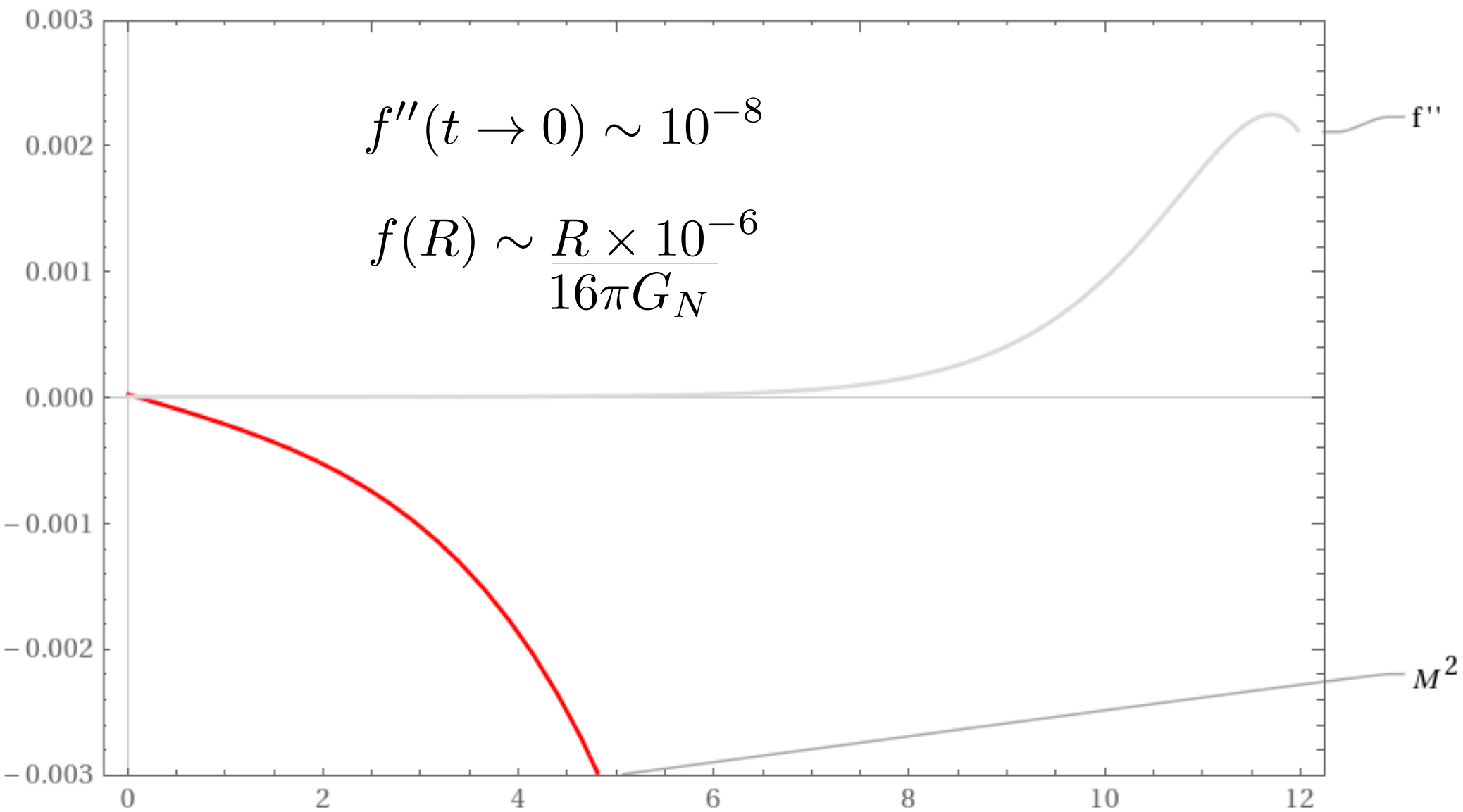




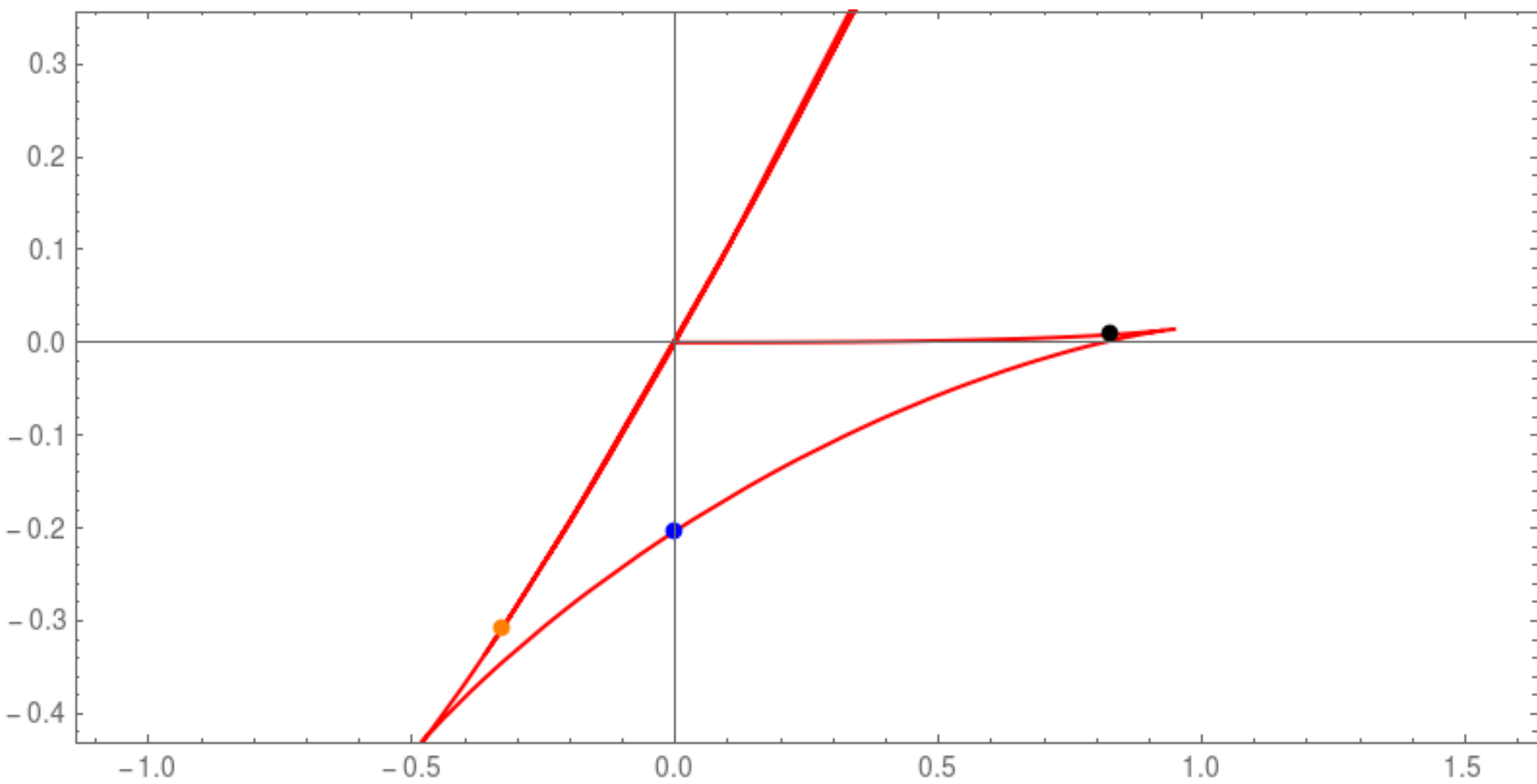


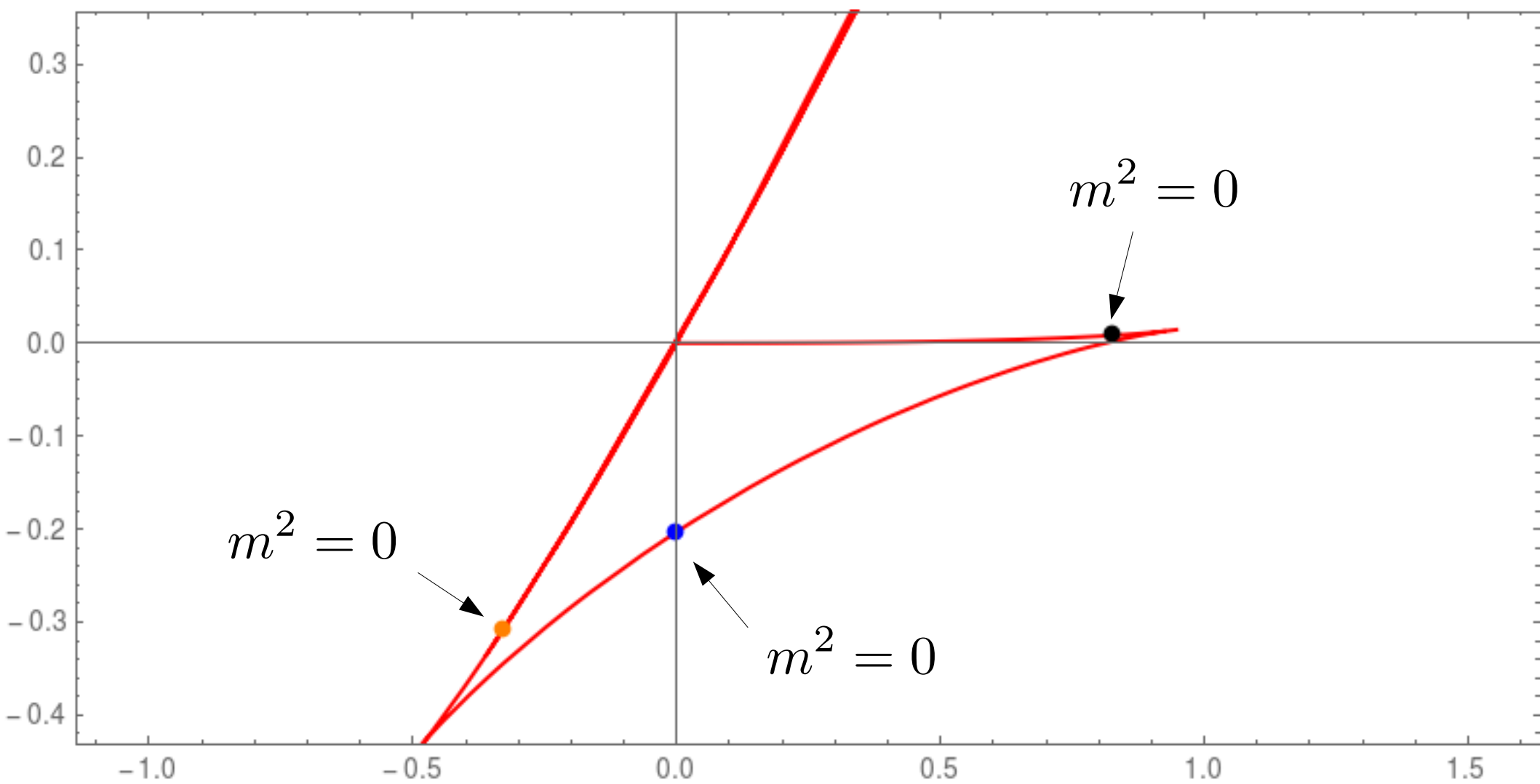


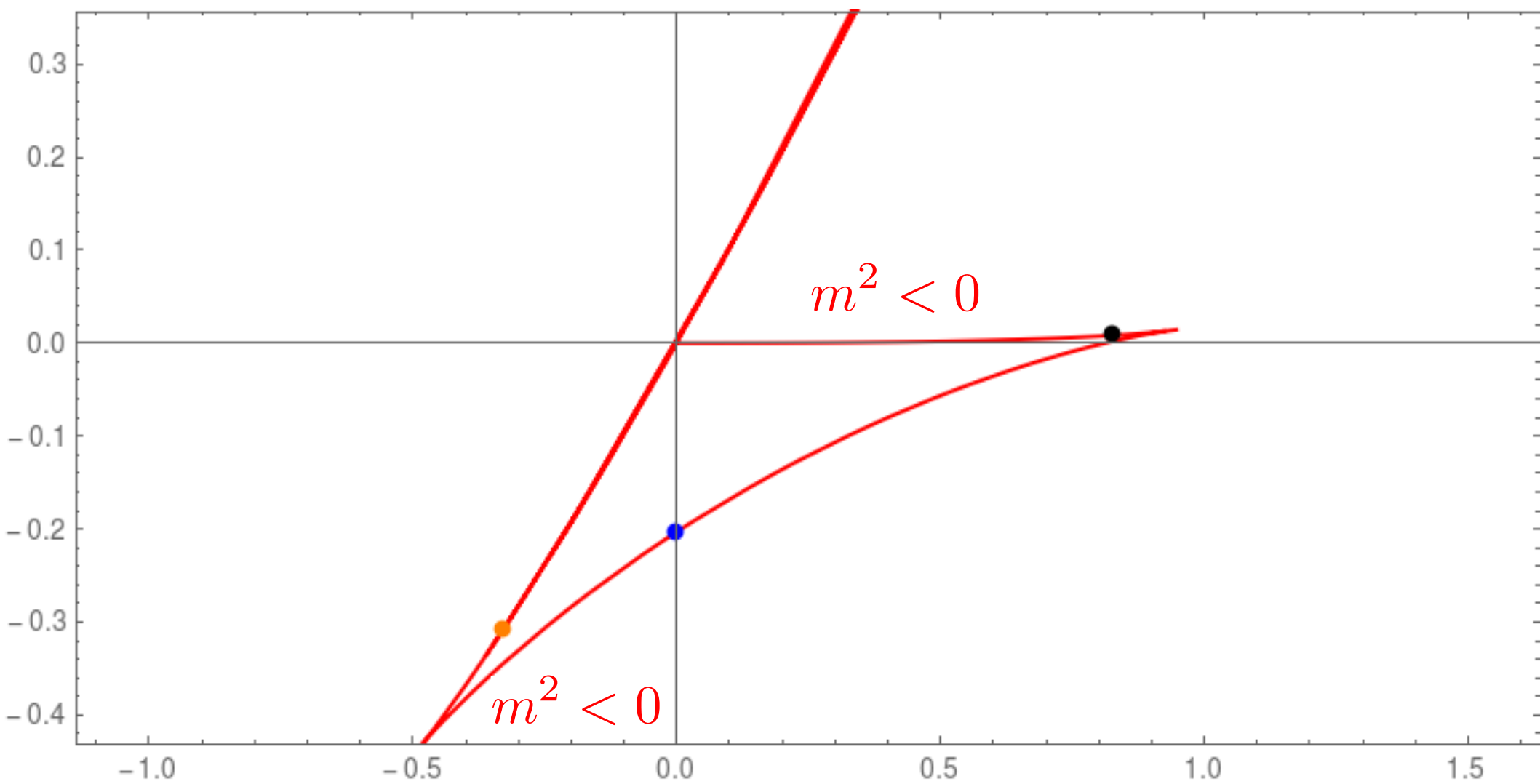


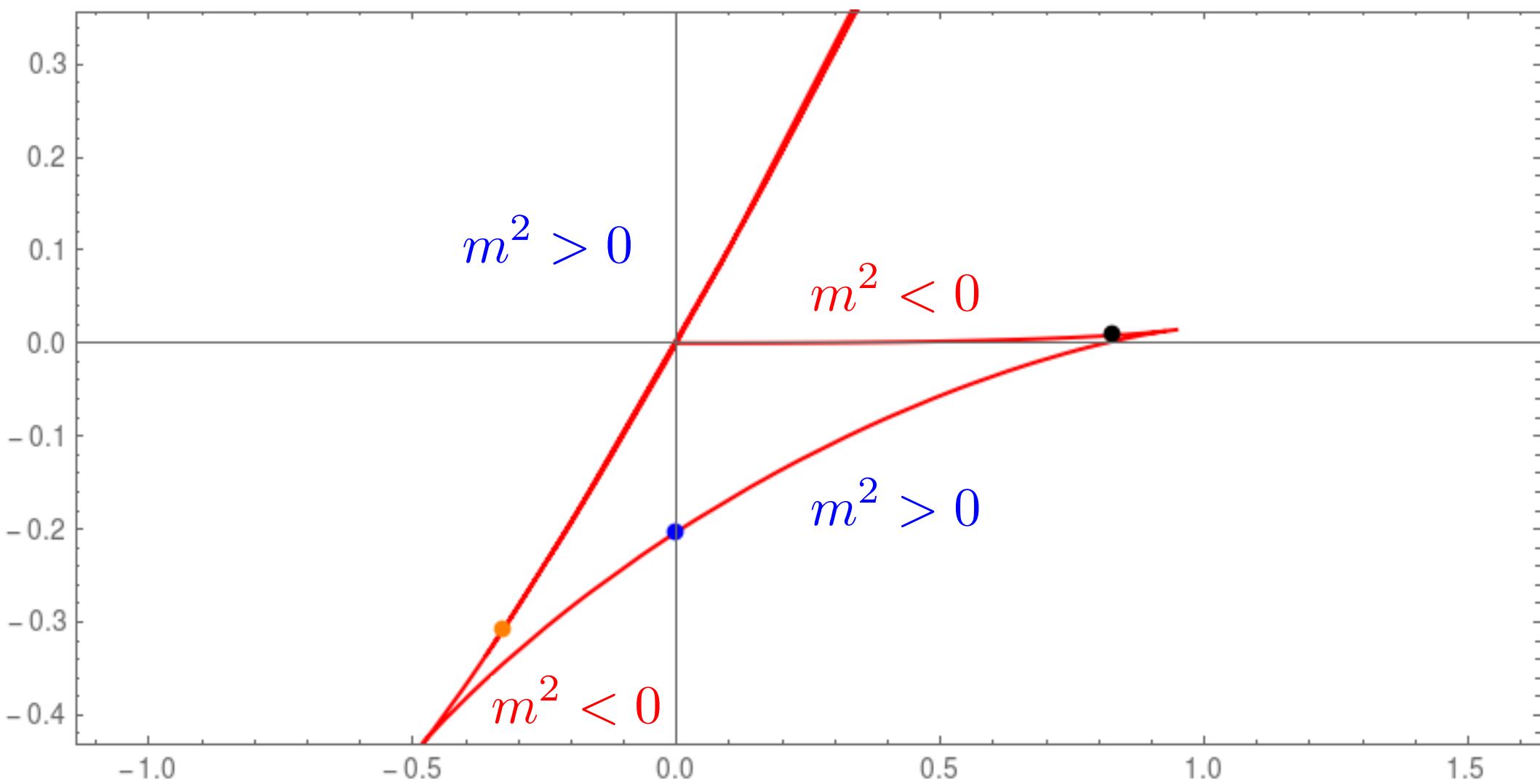


- **Inflation in Einstein frame $\Leftrightarrow$ linear Lagrangean in Jordan frame with an effective**
- **Still, it is Unstable...**
- **...as it should be, since the scale factor grows (nearly) exponentially**









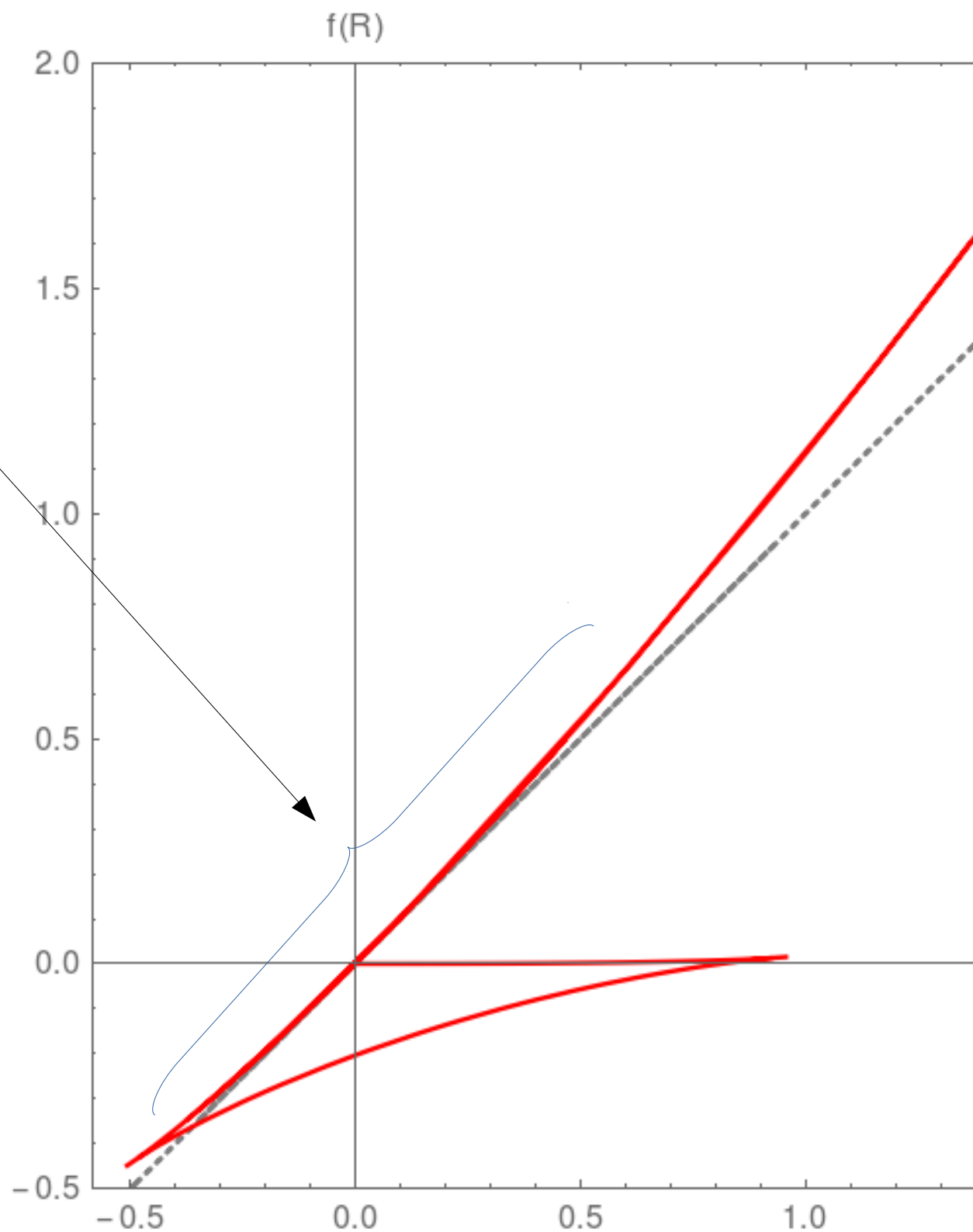
$$f(R) = R + \epsilon \Delta(R)$$

< 0 : *tachyons* !

$$\ddot{R}_1 - \nabla^2 R_1 - \frac{2\Delta'''}{\Delta''} \dot{T} \dot{R}_1 + \frac{1}{3\Delta''} \left(\frac{1}{\epsilon} - \Delta' \right) R_1 +$$

$$+ \frac{2\Delta'''}{\Delta''} \vec{\nabla} T \cdot \vec{\nabla} R_1 = \ddot{T} - \nabla^2 T - \frac{T\Delta' + \Delta}{3\Delta''}$$

$$f(R) = R + \epsilon \Delta(R)$$



- Initial instability is temporary
- Branches are separated by a change of sign of

$$f''(R)$$

Dynamical character

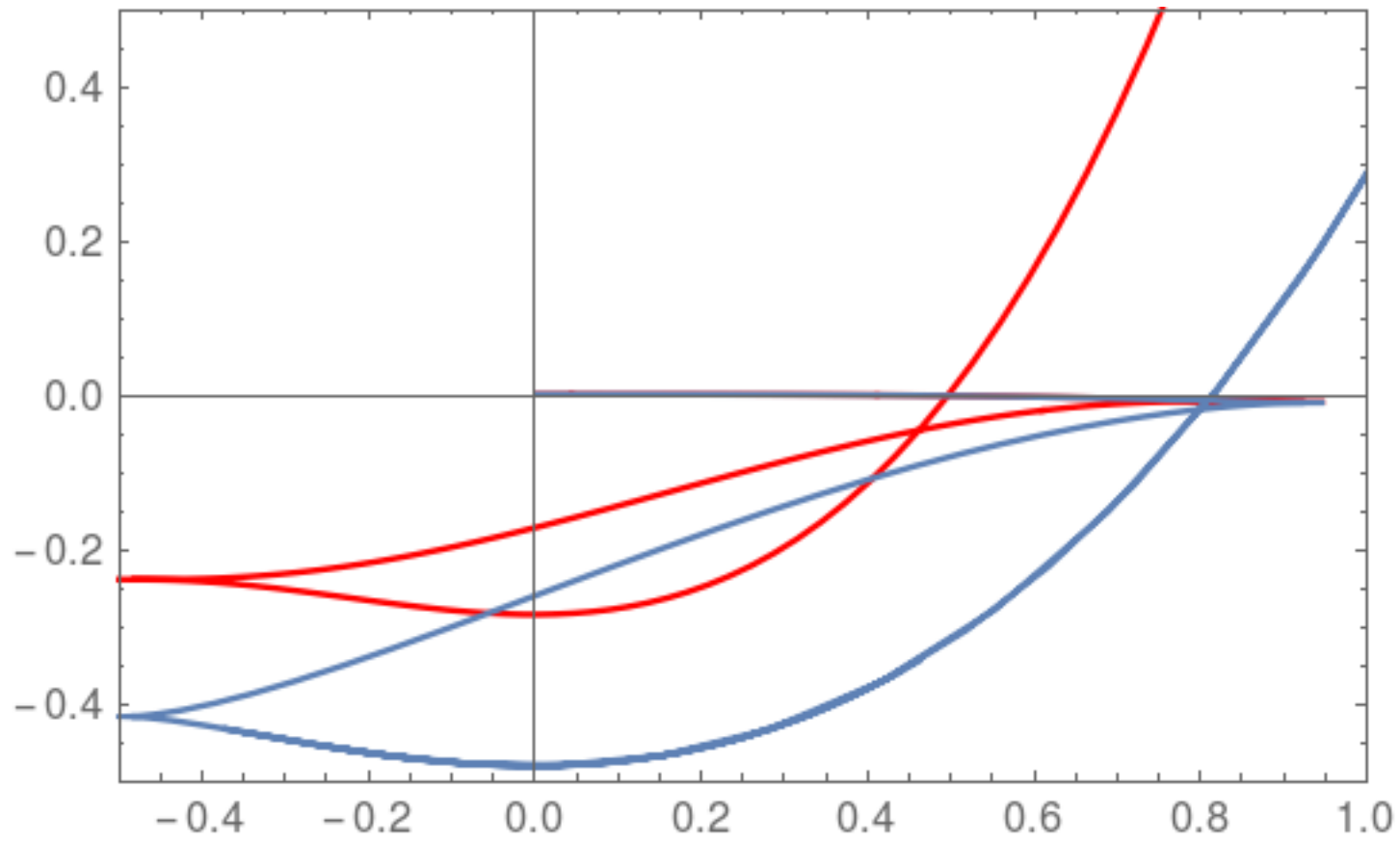
$$\frac{dV}{dR} \equiv \frac{f_R R - 2f}{3f_{RR}}$$

$$V(R) = \int \frac{f_R R - 2f}{3f_{RR}} \frac{dR}{dt} dt$$

Dynamical character

$$\frac{dV}{dR} \equiv \frac{f_R R - 2f}{3f_{RR}}$$

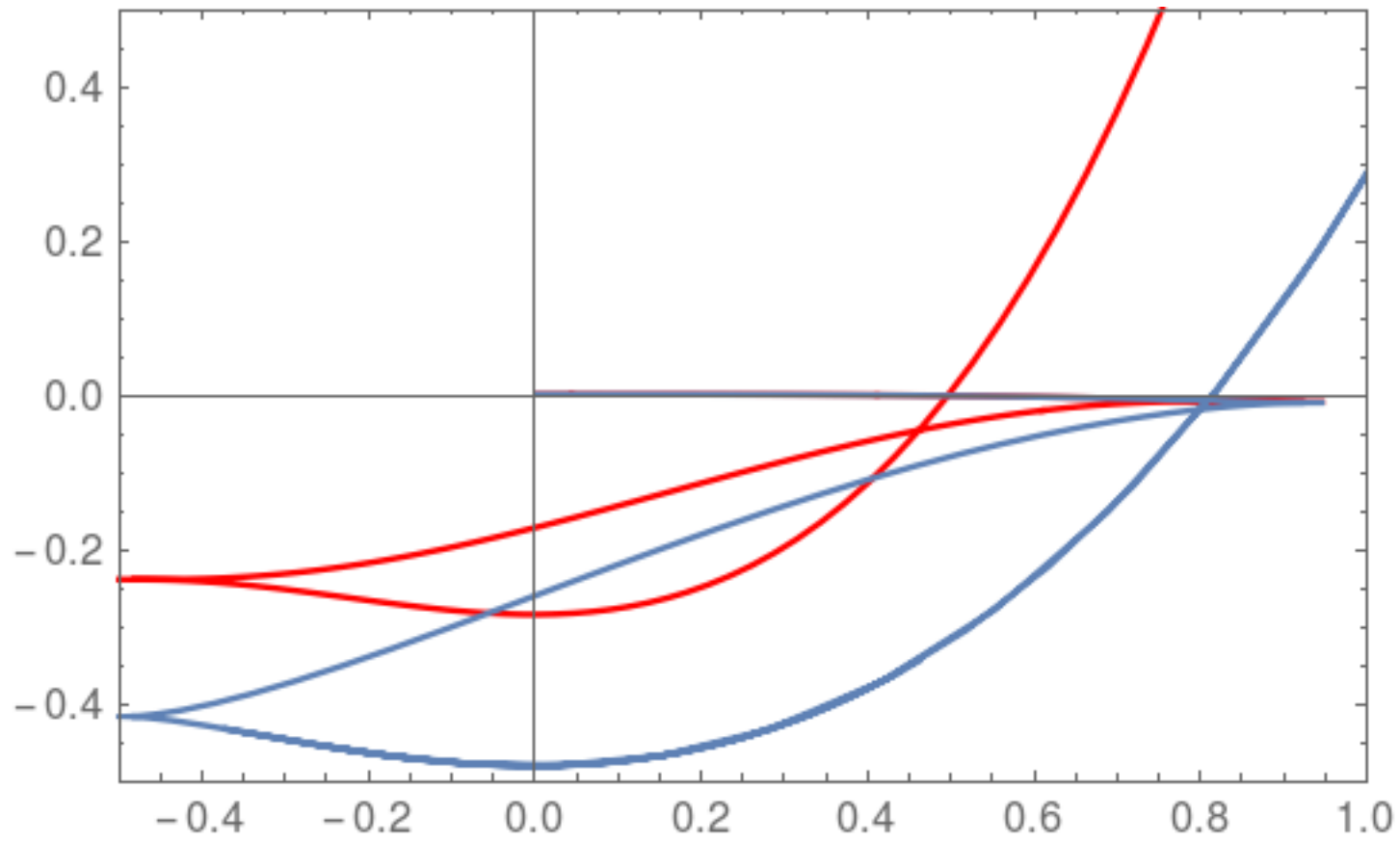
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Dynamical character

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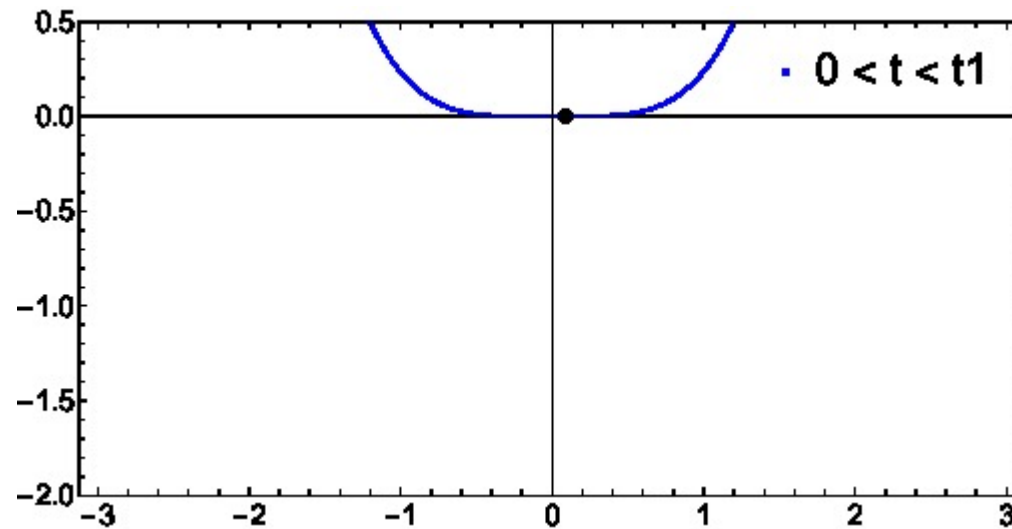
$$V_c(R, t) = \frac{1}{4}R^4 + \frac{u(t)}{2}R^2 + v(t)R + c$$



Dynamical character

$$\frac{dV}{dR} \equiv \frac{f_R R - 2f}{3f_{RR}}$$

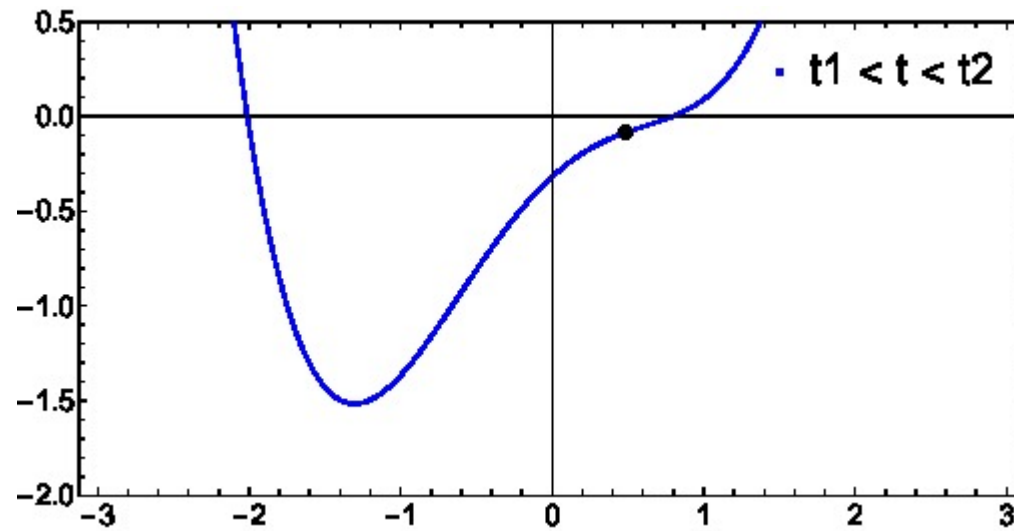
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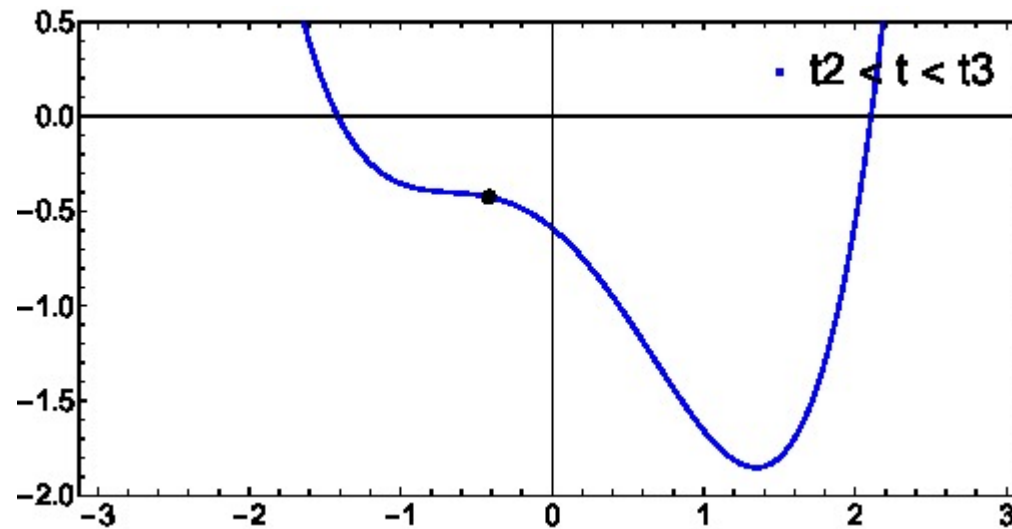
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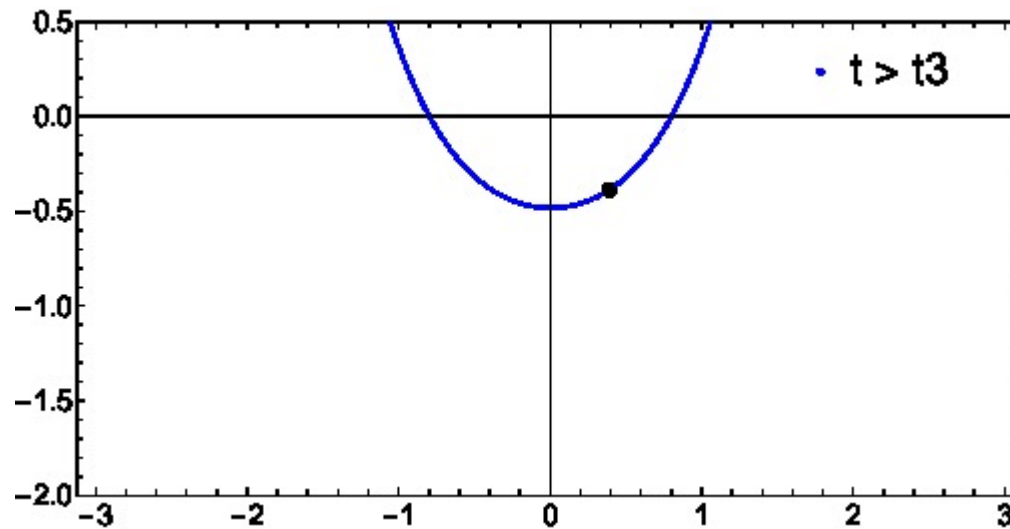
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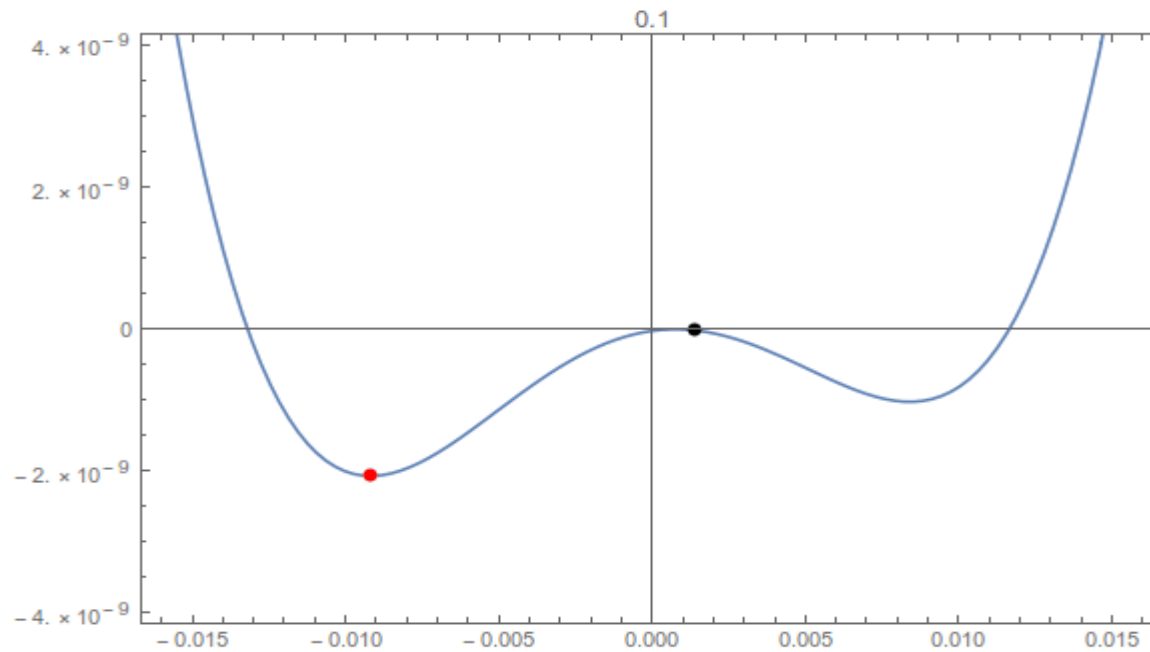
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$$V_c(R, t) = \frac{1}{4}R^4 + \frac{u(t)}{2}R^2 + v(t)R + c$$



Van der Waals Gas

$$\left(P + \frac{\alpha}{V^2}\right) \left(\frac{V}{N} - \beta\right) = R_G T$$

$$P_c \equiv \frac{\alpha}{27\beta^2} \quad V_c \equiv 3\beta \quad T_c \equiv \frac{8\alpha}{27\beta R_G}$$

$$\left(\tilde{P} + \frac{3}{\tilde{V}^2}\right) \left(\tilde{V} - \frac{1}{3}\right) = \frac{8}{3}\tilde{T}$$

$$\left(\tilde{P} + \frac{3}{\tilde{V}^2}\right) \left(\tilde{V} - \frac{1}{3}\right) = \frac{8}{3}\tilde{T}$$

$$p \equiv \tilde{P} - 1 \quad x \equiv \frac{1}{\tilde{V}} - 1 \quad t \equiv \tilde{T} - 1$$

$$x^3 + \left(\frac{8t + p}{3}\right)x + \left(\frac{8t - 2p}{3}\right) = 0$$

Equilibrium states

$$x^3 + \left(\frac{8t + p}{3} \right) x + \left(\frac{8t - 2p}{3} \right) = 0$$

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Equilibrium states

$$V_{\text{VdW}}(x) = \frac{x^4}{4} + \frac{u}{2}x^2 + vx + c$$

$$x^3 + \left(\frac{8t + p}{3} \right) x + \left(\frac{8t - 2p}{3} \right) = 0$$

Equilibrium states

$$V_{\text{VdW}}(x) = \frac{x^4}{4} + \frac{u}{2}x^2 + vx + c$$

$$t \equiv \frac{2u + v}{8}$$

$$p \equiv u - v$$

$$x \equiv R$$

$$x^3 + \left(\frac{8t+p}{3}\right)x + \left(\frac{8t-2p}{3}\right) = 0$$

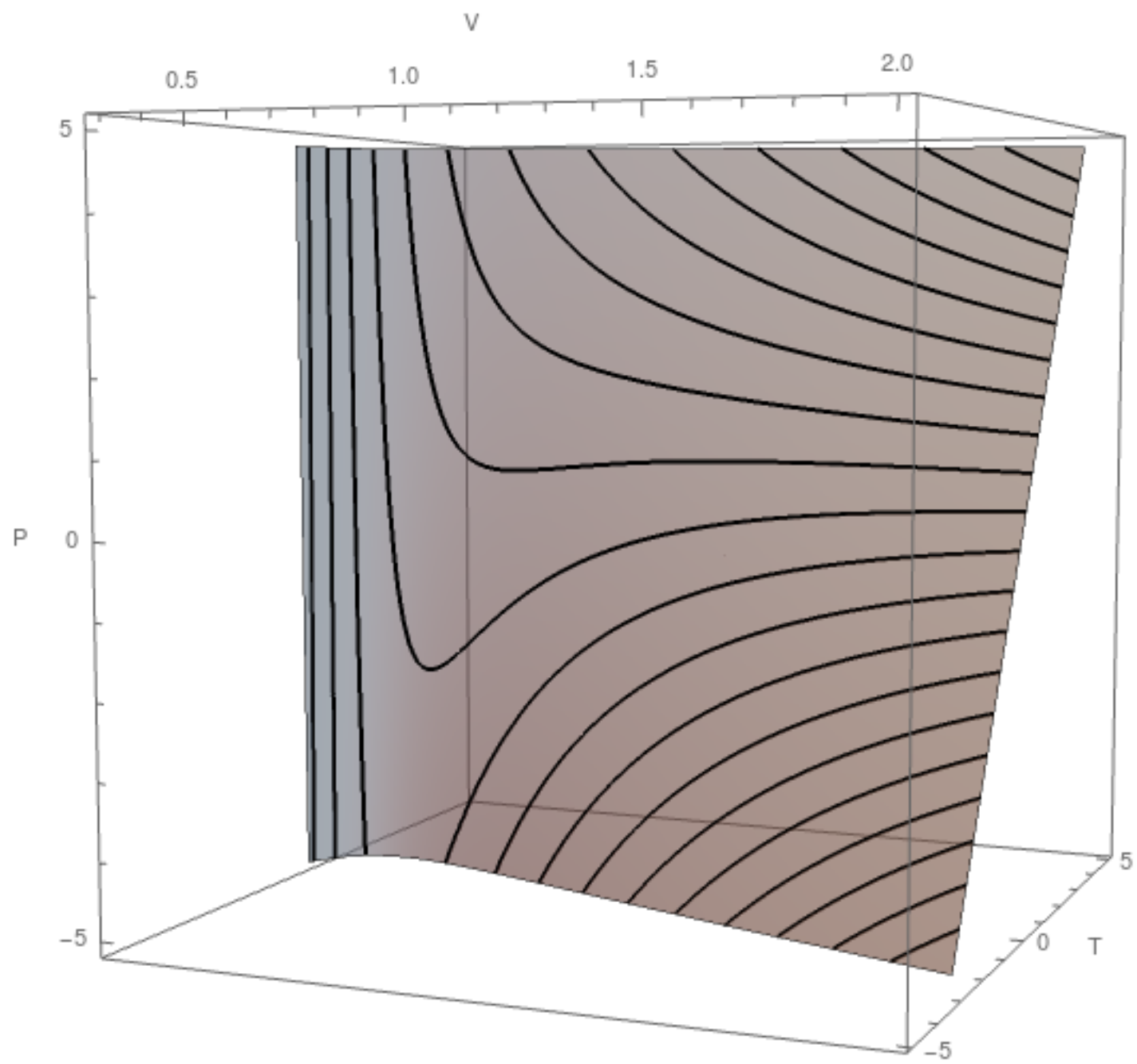
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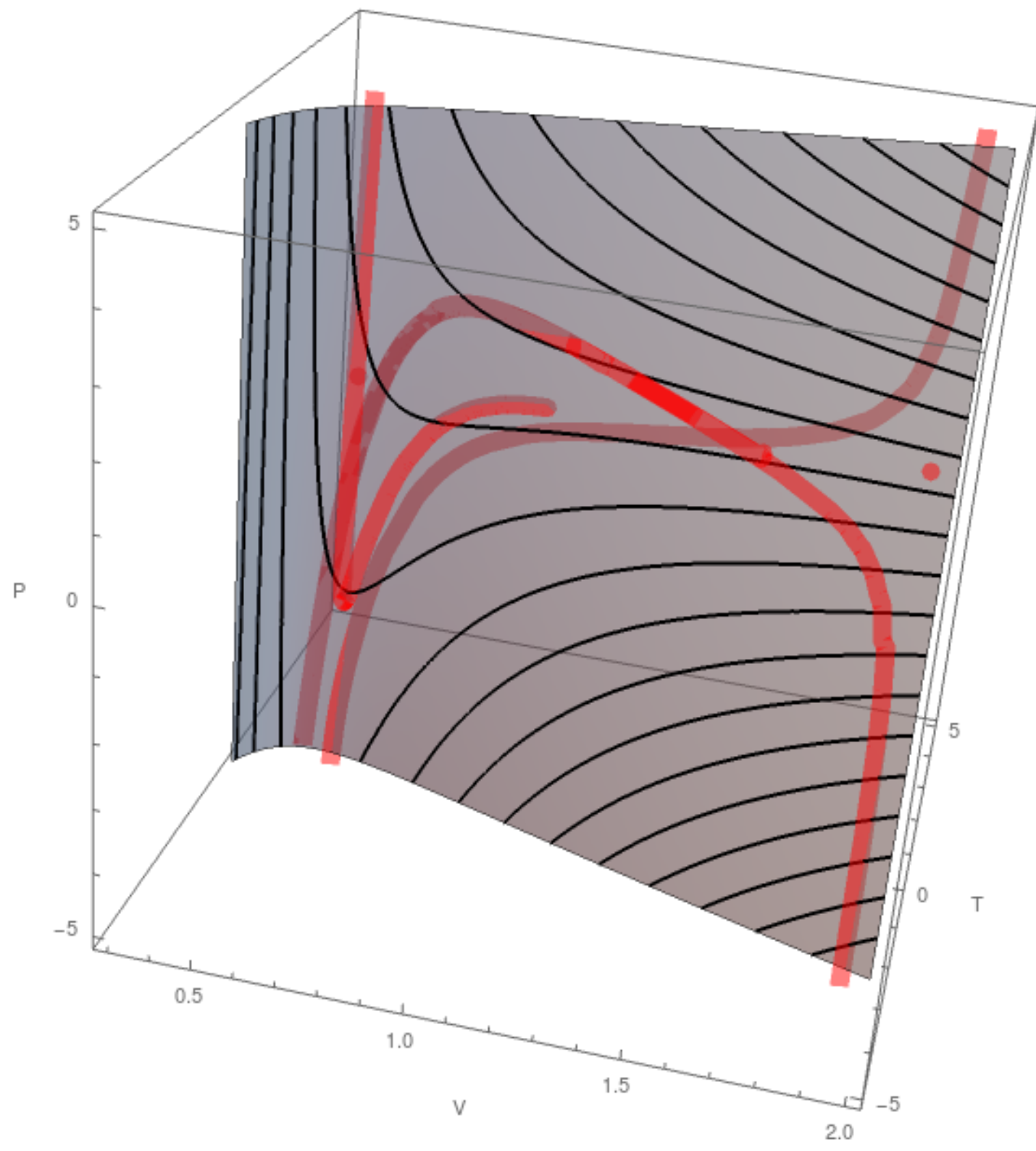
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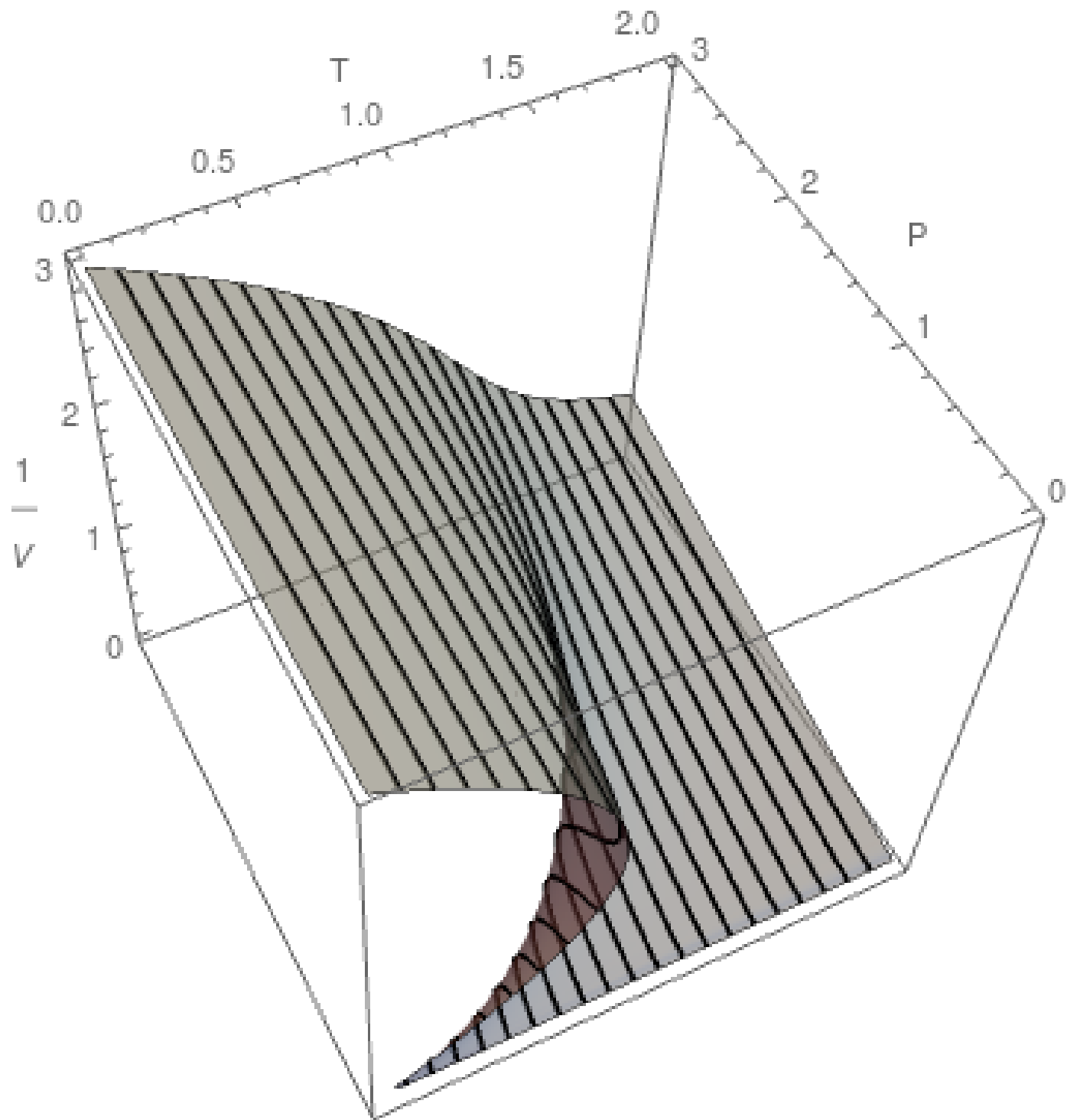
$$\tilde{T} \equiv \frac{2u+v}{8} + 1$$

$$\tilde{P} \equiv u - v + 1$$

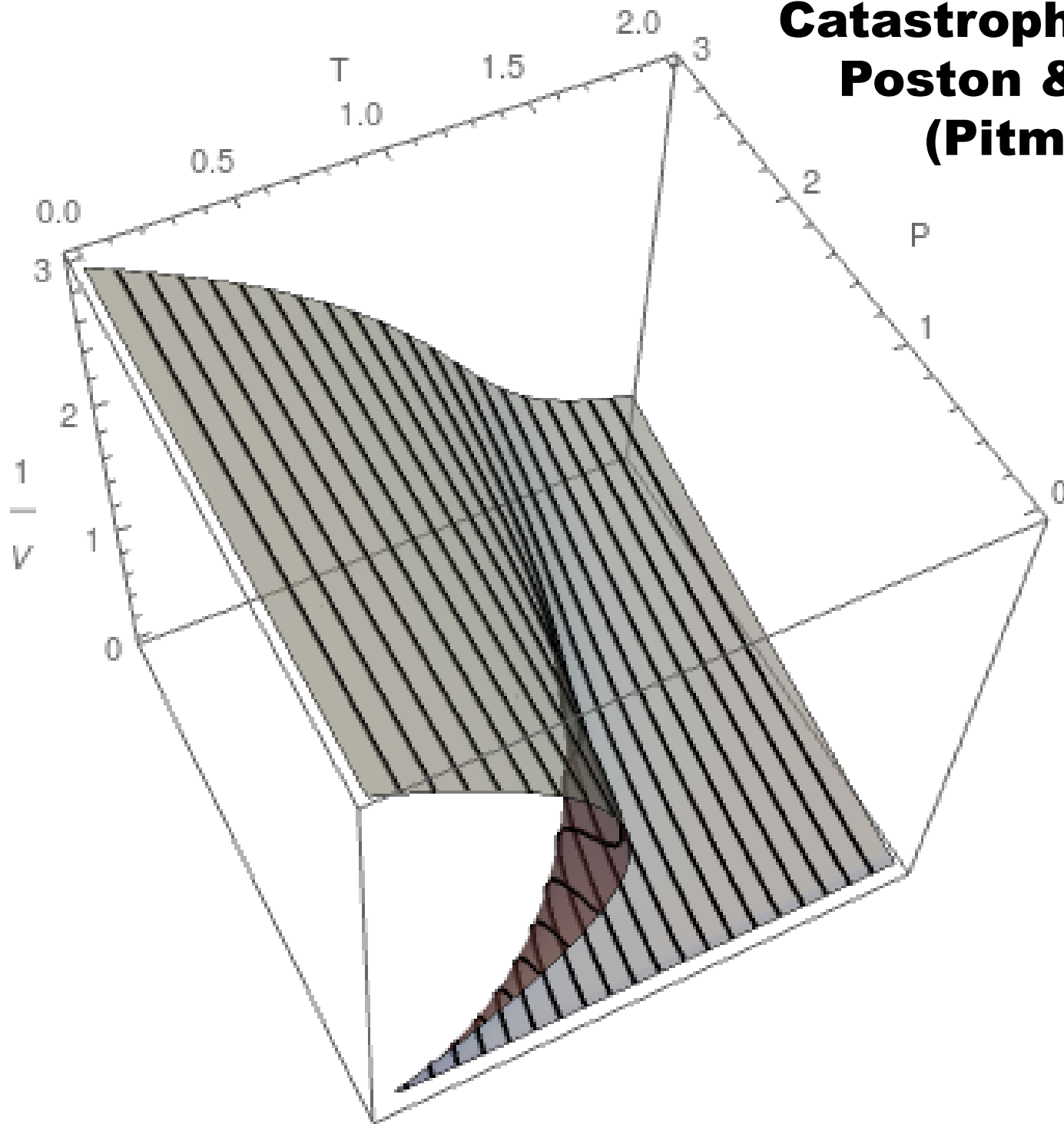
$$\tilde{V} \equiv \frac{1}{R+1}$$



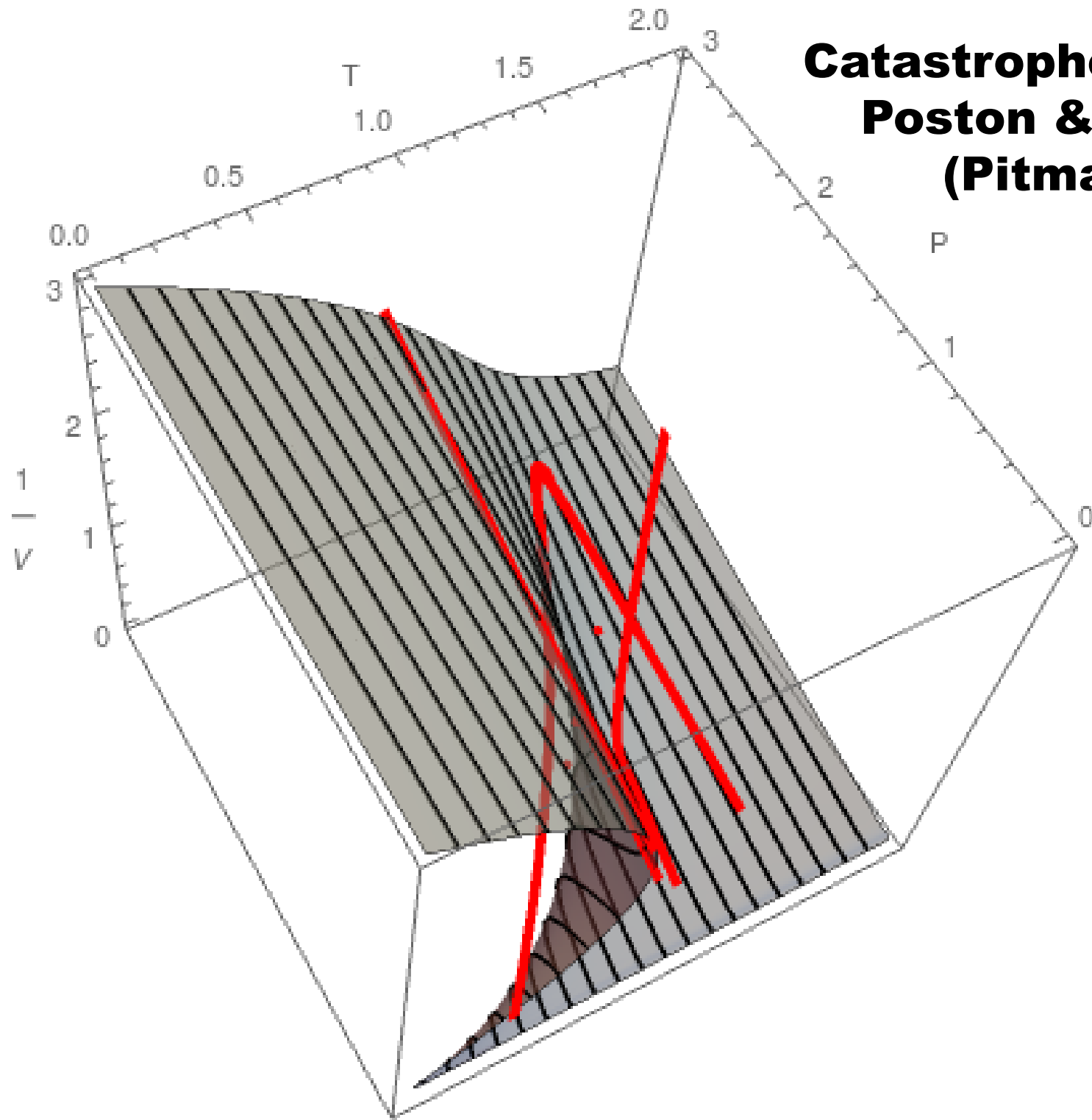




Catastrophe Theory Poston & Stewart (Pitman, 1978)



Catastrophe Theory Poston & Stewart (Pitman, 1978)



FINAL REMARKS

- Instability is not bad (as long as it is temporary!)
- can be either positive or negative !
- Even if $f''(R) < 0$, the outcome may be very different
- Recall that $f(R) \approx R$ but

$$\rho = 0 \forall t \qquad R(0) \neq 0$$



